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REFLECTION OF A SHORT NARROW LIGHT PULSE FROM A SCATTERING AND ABSORBING OCEAN*

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ABSTRACT

A very simple analytical expression for the shape of a pulse reflected from scattering and absorbing seawater is obtained. The resulting equation can be used for algorithms connected with the rapid assessment of optical water properties from remote platforms.

1.0 INTRODUCTION

Even though there is a multitude of publications devoted to the propagation of a light pulse through a scattering and absorbing medium like seawater (see Refs. in Dolin and Levin, 1991), very few of them are practical enough to be used in real-time detection algorithms, mainly because of the complexity of the resulting equations. Instead, inadequate and oversimplified expressions are coded in many real detection programs. In this article an attempt is made to derive a very simple analytical expression for the spacial-temporal shape of a light pulse propagating in seawater. The resulting equation has a very simple form and depends parametrically on the characteristics of the emitter and detector, as well as the inherent optical properties of water.

1.1 Emitter and Detector Parameters

First, let us specify the parameters of an emitter and detector. For simplicity, a Lambert-Gaussian detector is assumed. Such a detector adequately emulates the majority of real detectors. The sensitivity of this detector is regarded as Lambertian, *i.e.*, it is independent of the angle of incidence of light. The sensitivity of the detector surface declines with the distance ρ from the center of the detector according to the Gaussian law:

$$T_D(\rho) = \frac{k_D}{4\rho^2} \exp\left(-\frac{\pi\rho^2}{4\rho_D^2} \right), \qquad (1)$$

where ρ_D is the sensitivity radius which is defined by

$$\rho_D = \frac{2\pi}{k_D} \int_0^\infty T_D(\rho)\, \rho^2 d\rho, \qquad (2)$$

and k_D is the detection efficiency of the receiver

$$k_D = 2\pi \int_0^\infty T_D(\rho)\, \rho\, d\rho. \qquad (3)$$

We assume an emitter that is Gaussian both over the angle and over the distance from the center. It emits an infinitely short pulse represented by a temporal delta function. Such an assumption is mathematically convenient because the response from any arbitrary-shaped pulse can be calculated by mere convolution of the delta-shaped pulse response with the shape function of the real pulse (Morse and Feshbach, 1953). The energy density of the light pulse

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generated by this emitter is:

$$T_E(t,\rho,\mathbf{n}) = P_0\,A_E\,\exp\left(-\frac{\pi\rho^2}{4\rho_E^2} - 2\,\frac{1-\mathbf{n}\mathbf{n}_0}{D_\theta}\right)\delta(t)\ , \tag{4}$$

where ρ is the distance from the center of emitter in the plane which is orthogonal to the direction of the pulse propagation determined by the unit vector $\mathbf{n}_0$; $\mathbf{n}$ is a unit vector in the direction of propagation, t is the time, $\delta(t)$ is the Dirac delta function (Morse and Feshbach, 1953),

$$A_E = \left\{4\pi D_\theta \rho_E^2\left(1-\exp(-4/D_\theta)\right)\right\}^{-1} \cong 1/\left(4\pi D_\theta \rho_E^2\right) \tag{5}$$

is a normalization parameter adjusted in such a way that P_0 is the total energy of the pulse, ρ_E is the average radius of the pulse,

$$\rho_E = \int_0^\infty T_E(\rho)\,\rho^2\,d\rho \Big/ \int_0^\infty T_E(\rho)\,\rho\,d\rho\ , \tag{6}$$

D_θ is its angular dispersion,

$$D_\theta = \int_\Omega T_E(\rho)\sin^2(\theta)\,d\Omega \Big/ \int_\Omega T_E(\rho)\ d\Omega\,, \tag{7}$$

and P_0 is the pulse power:

$$P_0 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} 2\pi\rho\,d\rho \int_\Omega T_E(t,\rho,\mathbf{n})\,\mathbf{n}\mathbf{n}_0\,d\mathbf{n}\,. \tag{8}$$

1.2 Inherent Optical Properties of the Water

The inherent optical properties of the seawater are determined as follows. Here a is the absorption coefficient, b is the scattering coefficient, $c = a + b$ is the beam attenuation (or extinction) coefficient, $b_B = bB$ is the backscattering coefficient, $\omega_0 = b/c$ is the single scattering albedo, $p(\cos\chi)$ is the scattering phase function normalized such that

$$\frac{1}{2}\int_{-1}^{1} p(\mu)\,d\mu = 1\,,\quad \mu = \cos\chi\ , \tag{9}$$

and $B = 0.5\int_{-1}^{0} p(\mu)\,d\mu$ is the backscattering probability.

The scattering phase function is approximated by a superposition of two phase functions (Haltrin, 1985, 1988) as follows:

$$p(\mu) = (1-f)\,p_+(\mu) + f\,p_-(-\mu)\ . \tag{10}$$

One of them, $p_+(\mu)$, is a phase function which is strongly anisotropic in the forward direction, and the other, $p_-(\mu)$, is a phase function which is strongly anisotropic in the backward direction. These functions can be derived, for example, from Petzold's phase functions (see Haltrin, 1997).

2.0 COMPUTATIONAL APPROACH

2.1 Initial Equations

The starting equation is the time-dependent transport equation for the light radiance $L(t,\mathbf{r},\mathbf{n})$ of the pulse

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \mathbf{n}\nabla_{\mathbf{r}} + c\right)L(t,\mathbf{r},\mathbf{n}) = \frac{b}{4\pi}\int_\Omega L(t,\mathbf{r},\mathbf{n}')\,p(\mathbf{n}\mathbf{n}')\,d\mathbf{n}'\ , \tag{11}$$

where $\mathbf{r}=(x,y,z)=(\rho,z)$ is the spatial coordinate. Both Cartesian and cylindrical coordinates are used here. In both coordinate systems the $0z$ axis is directed along the propagation path, Cartesian x, y, and cylindrical ρ coordinates lie in the orthogonal plane, t is the time, $\mathbf{n}=(\sin\theta\cos\varphi,\,\sin\theta\sin\varphi,\,\cos\theta)$ is the scattering unit vector in the direction of propagation, θ and φ are the zenith and azimuth angles, $\mathbf{n}_0$ is the unit vector in the direction of propagation, Ω is the total solid angle, and v is the velocity of light in seawater. All angles here are determined inside the body of water.

The boundary condition just below the surface is expressed as:

$$L(t,\mathbf{r},\mathbf{n})\big|_{z=0} \equiv L_0(t,\rho,\mathbf{n}) = P_0\,A_E\,\exp\left(-\frac{\pi\rho^2}{4\rho_E^2}-2\frac{1-\mathbf{n}\mathbf{n}_0}{D_\theta}\right)\delta(t). \tag{12}$$

To solve Eqn. (11), the double-sided Laplace transform (Morse and Feshbach, 1953) of radiance is used. It is defined by the following equations:

$$L(t,\mathbf{r},\mathbf{n}) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} L_p(\mathbf{r},\mathbf{n})\,e^{pt}\,dp, \quad L_p(\mathbf{r},\mathbf{n}) = \int_0^{\infty} L(t,\mathbf{r},\mathbf{n})\,e^{-pt}\,dt. \tag{13}$$

With the introduction of a new parameter $\varepsilon = c + p/v$, the Laplace transform of Eqn. (11) can be rewritten as

$$(\mathbf{n}\nabla_r+\varepsilon)\,L_p(\mathbf{r},\mathbf{n}) = \frac{b}{4\pi}\int_\Omega L_p(\mathbf{r},\mathbf{n}')\,p(\mathbf{n}\mathbf{n}')\,d\mathbf{n}', \tag{14}$$

with the boundary condition

$$L_p(\mathbf{r},\mathbf{n})\big|_{z=0} = P_0\,A_E\,\exp\left(-\frac{\pi\rho^2}{4\rho_E^2}-2\frac{1-\mathbf{n}\mathbf{n}_0}{D_\theta}\right). \tag{15}$$

It is convenient for further calculations to split the light radiance into two parts, forward (L_{p1}) and backward (L_{p2}) ones:

$$L_p(\mathbf{r},\mathbf{n}) = \begin{cases} L_{p1}(\mathbf{r},\,\mathbf{n}) = L_p(\mathbf{r},\,\mathbf{n}), & \mathbf{n}\in\Omega_+, \\ L_{p2}(\mathbf{r},\,\mathbf{n}) = L_p(\mathbf{r},-\mathbf{n}), & \mathbf{n}\in\Omega_+. \end{cases} \tag{16}$$

Equation (14) in terms of Eqn. (16) can be represented as the following system of equations for the forward and backward radiances:

$$\left.\begin{aligned} (\mathbf{n}\nabla_r+\varepsilon)\,L_{p1}(\mathbf{r},\mathbf{n}) &= \frac{b}{4\pi}\int_{\Omega_+} L_{p1}(\mathbf{r},\mathbf{n})\,p(\mathbf{n}\mathbf{n}')\,d\mathbf{n}' + \frac{b}{4\pi}\int_{\Omega_+} L_{p2}(\mathbf{r},\mathbf{n})\,p(-\mathbf{n}\mathbf{n}')\,d\mathbf{n}', \\ (-\mathbf{n}\nabla_r+\varepsilon)\,L_{p2}(\mathbf{r},\mathbf{n}) &= \frac{b}{4\pi}\int_{\Omega_+} L_{p1}(\mathbf{r},\mathbf{n})\,p(-\mathbf{n}\mathbf{n}')\,d\mathbf{n}' + \frac{b}{4\pi}\int_{\Omega_+} L_{p2}(\mathbf{r},\mathbf{n})\,p(\mathbf{n}\mathbf{n}')\,d\mathbf{n}'. \end{aligned}\right\} \tag{17}$$

By using Eqn. (10) for the phase function of scattering, Eqns. (17) may be rewritten as:

$$\begin{aligned} (\mathbf{n}\nabla_r+\varepsilon)\,L_{p1}(\mathbf{r},\mathbf{n}) &= Q_1(\mathbf{r},\mathbf{n})+\Delta_1(\mathbf{r},\mathbf{n}), \\ (-\mathbf{n}\nabla_r+\varepsilon)\,L_{p2}(\mathbf{r},\mathbf{n}) &= Q_2(\mathbf{r},\mathbf{n})+\Delta_2(\mathbf{r},\mathbf{n}), \end{aligned} \tag{18}$$

where

$$Q_1(\mathbf{r},\mathbf{n}) = \frac{b\,(1-f)}{4\pi}\int_{\Omega_+} L_{p1}(\mathbf{r},\mathbf{n})\,p_+(\mathbf{n}\mathbf{n}')\,d\mathbf{n}' + \frac{b\,f}{4\pi}\int_{\Omega_+} L_{p2}(\mathbf{r},\mathbf{n})\,p_-(\mathbf{n}\mathbf{n}')\,d\mathbf{n}', \tag{19}$$

$$Q_2(\mathbf{r},\mathbf{n}) = \frac{b\,(1-f)}{4\pi}\int_{\Omega_+} L_{p2}(\mathbf{r},\mathbf{n})\,p_+(\mathbf{n}\mathbf{n}')\,d\mathbf{n}' + \frac{b\,f}{4\pi}\int_{\Omega_+} L_{p1}(\mathbf{r},\mathbf{n})\,p_-(\mathbf{n}\mathbf{n}')\,d\mathbf{n}', \tag{20}$$

$$\Delta_1(\mathbf{r},\mathbf{n}) = \frac{b\,f}{4\pi}\int_{\Omega_+} L_{p1}(\mathbf{r},\mathbf{n})\,p_-(-\mathbf{nn}')\,d\mathbf{n}' + \frac{b(1-f)}{4\pi}\int_{\Omega_+} L_{p2}(\mathbf{r},\mathbf{n})\,p_+(-\mathbf{nn}')\,d\mathbf{n}', \qquad (21)$$

$$\Delta_2(\mathbf{r},\mathbf{n}) = \frac{b\,f}{4\pi}\int_{\Omega_+} L_{p2}(\mathbf{r},\mathbf{n})\,p_-(-\mathbf{nn}')\,d\mathbf{n}' + \frac{b(1-f)}{4\pi}\int_{\Omega_+} L_{p1}(\mathbf{r},\mathbf{n})\,p_+(-\mathbf{nn}')\,d\mathbf{n}'. \qquad (22)$$

2.2 Small-Angle Approximation

Let us choose the phase function components p_+ and p_- in such a manner that their *tails* in the backward hemisphere are exponentially small, so that

$$|\Delta_1| << |Q_1| \text{ and } |\Delta_2| << |Q_2|. \qquad (23)$$

The resulting phase function from Eqn.(10) with its parts satisfying the inequalities above still gives us a very satisfactory approximation for a realistic ocean phase function (Haltrin, 1984).

Let us solve Eqns. (18) in the small-angle approximation (Wells, 1982; Walker, 1987; Arnush, 1972; Dolin and Levin, 1991) with the phase functions described above. We should also make the following simplifications that are typical for the small-angle approximation:

$$\left.\begin{array}{c} \mathbf{n} = n_z + \mathbf{s}, \quad n_z \cong \left(0,\ 0,\ 1-\mathbf{s}^2/2\right), \quad \mathbf{n}\nabla_r \cong \left(1-\frac{1}{2}\mathbf{s}^2\right)\frac{\partial}{\partial z} + \mathbf{s}\nabla_\rho, \quad p_\pm(\mathbf{nn}') \to p_\pm(|\mathbf{s}'|), \\ \mathbf{nn}' = 1-(\mathbf{n}-\mathbf{n}')^2/2, \quad \mathbf{n}-\mathbf{n}' = \mathbf{s}', \quad \mathbf{n}' = \mathbf{n}+(\mathbf{n}'-\mathbf{n}) = 1+(\mathbf{s}-\mathbf{s}'), \\ L_{pi}(\mathbf{r},\mathbf{n}) \to L_{pi}(z,\rho,\mathbf{s}), \quad L_{pi}(\mathbf{r},\mathbf{n}') \to L_{pi}(z,\rho,\mathbf{s}-\mathbf{s}'). \end{array}\right\} \qquad (24)$$

Now we have the approximate system of equations for the Laplace transforms of radiances:

$$\left.\begin{array}{l} \left[\ \left(1-\frac{1}{2}\mathbf{s}^2\right)\frac{\partial}{\partial z} + \mathbf{s}\nabla_\rho + \varepsilon\right] L_{p1}(z,\rho,\mathbf{s}) = Q_1(z,\rho,\mathbf{s}), \\ \left[-\left(1-\frac{1}{2}\mathbf{s}^2\right)\frac{\partial}{\partial z} - \mathbf{s}\nabla_\rho + \varepsilon\right] L_{p2}(z,\rho,\mathbf{s}) = Q_2(z,\rho,\mathbf{s}). \end{array}\right\} \qquad (25)$$

The Fourier transform of radiance in the plane that is orthogonal to the direction of the pulse propagation is:

$$L_{pi}(z,\rho,\mathbf{s}) = \iint d\mathbf{k}\,d\mathbf{q}\,F_{pi}(z,\mathbf{k},\mathbf{q})\,e^{-i\mathbf{k}\rho - i\mathbf{qs}}, \qquad (26)$$

$$F_{pi}(z,\mathbf{k},\mathbf{q}) = \frac{1}{(2\pi)^4}\iint d\rho\,d\mathbf{s}\,L_{pi}(z,\rho,\mathbf{s})\,e^{i\mathbf{k}\rho + i\mathbf{qs}}. \qquad (27)$$

Now we have the following system of equations for the Laplace-Fourier transforms of the forward and backward radiances:

$$\left.\begin{array}{l} \left[\ \left(1-\frac{1}{2}\mathbf{s}^2\right)\frac{\partial}{\partial z} - \mathbf{k}\frac{\partial}{\partial \mathbf{q}} + \varepsilon\right] F_{p1}(z,\mathbf{k},\mathbf{q}) = V_+(\mathbf{q})\,F_{p1}(z,\mathbf{k},\mathbf{q}) + V_-(\mathbf{q})\,F_{p2}(z,\mathbf{k},\mathbf{q}), \\ \left[-\left(1-\frac{1}{2}\mathbf{s}^2\right)\frac{\partial}{\partial z} + \mathbf{k}\frac{\partial}{\partial \mathbf{q}} + \varepsilon\right] F_{p2}(z,\mathbf{k},\mathbf{q}) = V_+(\mathbf{q})\,F_{p2}(z,\mathbf{k},\mathbf{q}) + V_-(\mathbf{q})\,F_{p1}(z,\mathbf{k},\mathbf{q}), \end{array}\right\} \qquad (28)$$

where

$$V_\pm(\mathbf{q}) = \frac{b(1-f)}{4\pi}\int p_\pm(|\mathbf{s}|)\,e^{i\mathbf{qs}}\,d\mathbf{s} = \frac{b(1-f)}{2}\int_0^{\theta_{\max}} J_0(q\theta)\,p_\pm(\theta)\,d\theta\,, \qquad (29)$$

and J_0 is the zero-order Bessel function. Retaining only terms proportional to $\mathbf{q}^2$, we get:

$$\left.\begin{aligned} V_+(\mathbf{q}) &\cong b(1-f) - b\left\langle\theta^2\right\rangle\mathbf{q}^2/4, \\ V_-(\mathbf{q}) &\cong b\,f - b\left\langle(\pi-\theta)^2\right\rangle\mathbf{q}^2/4, \end{aligned}\right\} \tag{30}$$

where the angular brackets $\langle ... \rangle$ denote averaging over the phase function given by Eqn. (10) according to the rule:

$$\left\langle x(\mu)\right\rangle = \frac{1}{2}\int_{-1}^{1} p(\mu)\,x(\mu)\,d\mu \quad . \tag{31}$$

Equations (28) for the light pulse components should satisfy the following boundary condition:

$$F_{p1}(z,\mathbf{k},\mathbf{q})\Big|_{z=0} = F_{p0}(\mathbf{k},\mathbf{q}) = P_0 \exp\left(-\frac{\rho_E^2\,\mathbf{k}^2}{\pi} - \frac{D_\theta\,\mathbf{q}^2}{4}\right). \tag{32}$$

Now we can estimate the terms in the left part of Eqn. (30) which are proportional to $\mathbf{s}^2$:

$$\frac{1}{2}\mathbf{s}^2\frac{\partial}{\partial z}L_i \sim \left\langle\theta^2\right\rangle c\,L_i. \tag{33}$$

At the same time our corrections due to the phase function in Eqn (27) have the following order of magnitude:

$$\frac{b}{4}\left\langle\theta_\pm^2\right\rangle q^2\,L_i \sim b\left\langle\theta^2\right\rangle\frac{1}{\left\langle\theta^2\right\rangle}L_i \sim b\,L_i. \tag{34}$$

So, if the condition $\left\langle\theta^2\right\rangle << \omega_0$ holds, all terms which are proportional to $\mathbf{s}^2$ in the left part of Eqn. (28) may be neglected. With this in mind, Eqns. (24) acquire the following form:

$$\left.\begin{aligned} &\left(\frac{\partial}{\partial z} - \mathbf{k}\frac{\partial}{\partial\mathbf{q}} + \alpha + \beta_1\,\mathbf{q}^2\right)F_{p1}(z,\mathbf{k},\mathbf{q}) - \left(bf - \beta_2\,\mathbf{q}^2\right)F_{p2}(z,\mathbf{k},\mathbf{q}) = 0\,, \\ &-(bf - \beta_2\,\mathbf{q}^2)\,F_{p1}(z,\mathbf{k},\mathbf{q}) + \left(-\frac{\partial}{\partial z} + \mathbf{k}\frac{\partial}{\partial\mathbf{q}} + \alpha + \beta_1\,\mathbf{q}^2\right)F_{p2}(z,\mathbf{k},\mathbf{q}) = 0\,, \end{aligned}\right\} \tag{35}$$

where $\alpha = a + bf + p/v$, $\beta_1 = b < \theta^2 > /4$ and $\beta_2 = b < (\pi-\theta)^2 > /4$, and the boundary condition for Eqns. (35) is given by Eqn. (32).

Next, let us represent the downward pulse radiance as a sum of the unscattered part F_{p1}^Q (the source) and the scattered part F_{p1}^s:

$$F_{p1}(z,\mathbf{k},\mathbf{q}) = F_{p1}^Q(z,\mathbf{k},\mathbf{q}) + F_{p1}^s(z,\mathbf{k},\mathbf{q}). \tag{36}$$

The backward pulse radiance consists only of scattered radiation $F_{p2}(z,\mathbf{k},\mathbf{q}) = F_{p2}^s(z,\mathbf{k},\mathbf{q})$. The unscattered forward pulse radiance F_{p1}^Q satisfies the following propagation equation:

$$\left(\frac{\partial}{\partial z} - \mathbf{k}\frac{\partial}{\partial\mathbf{q}} + \alpha + \beta_1\,\mathbf{q}^2\right)F_{p1}^Q(z,\mathbf{k},\mathbf{q}) = 0\,, \tag{37}$$

with the same condition on the boundary given by Eqn. (29). The solution to Eqn. (37) is given by the expression:

$$F_{p1}^{Q}(z,\mathbf{k},\mathbf{q}) = F_p^0(\mathbf{k},\mathbf{q}+\mathbf{k}z)\exp\left[-\alpha z - \beta_1\int_0^z(\mathbf{q}+\mathbf{k}\eta)^2\,d\eta\right]. \tag{38}$$

This expression can also be represented by the following analytical formula:

$$F_{p1}^{Q}(z,\mathbf{k},\mathbf{q}) = P_0\exp\left[-\frac{\rho_E^2\,\mathbf{k}^2}{\pi} - \frac{D_\theta\,\mathbf{q}^2}{4} - \left(\alpha + \frac{D_\theta}{2}\mathbf{kq} + \beta_1\mathbf{q}^2\right)z - \left(\frac{D_\theta}{4}\mathbf{k}^2 + \beta_1\mathbf{kq}\right)z^2 - \beta_1\frac{\mathbf{k}^2}{3}z^3\right]. \tag{39}$$

The scattered parts of the forward and backward radiances of the pulse are satisfied by the following equations:

$$\left.\begin{aligned}\left(-\frac{\partial}{\partial z}+\mathbf{k}\frac{\partial}{\partial\mathbf{q}}+\alpha+\beta_1\,\mathbf{q}^2\right)F_{p2}^{s}(z,\mathbf{k},\mathbf{q}) &= (bf-\beta_2\,\mathbf{q}^2)\,F_{p1}^{Q}(z,\mathbf{k},\mathbf{q}),\\ \left(\frac{\partial}{\partial z}-\mathbf{k}\frac{\partial}{\partial\mathbf{q}}+\alpha+\beta_1\,\mathbf{q}^2\right)F_{p1}^{s}(z,\mathbf{k},\mathbf{q}) &= (bf-\beta_2\,\mathbf{q}^2)\,F_{p2}^{s}(z,\mathbf{k},\mathbf{q}),\end{aligned}\right\} \tag{40}$$

with the boundary conditions

$$F_{p1}^{s}(z,\mathbf{k},\mathbf{q})\Big|_{z=0} = 0,\quad \lim_{z\to\infty}F_{pi}^{s}(z,\mathbf{k},\mathbf{q}) = 0,\quad i=1,2. \tag{41}$$

The next step, according to the method given by Both (1929), Snyder (1949) and Romanova (1968), involves the following substitutions:

$$\mathbf{q}=\mathbf{g}-\mathbf{k}z,\quad F_{pi}^{s}(z,\mathbf{k},\mathbf{q}) = F_{pi}^{s}(z,\mathbf{k},\mathbf{g}-\mathbf{k}z)\equiv\Phi_{pi}^{s}(z,\mathbf{k},\mathbf{g}), \tag{42}$$

which convert Eqns. (40) into the following one-dimensional system of two equations:

$$\left.\begin{aligned}\left[-\frac{d}{dz}+\alpha+\beta_1(\mathbf{g}-\mathbf{k}z)^2\right]\Phi_{p2}^{s}(z,\mathbf{k},\mathbf{g}) &= \left[bf-\beta_2\,(\mathbf{g}-\mathbf{k}z)^2\right]F_{p1}^{Q}(z,\mathbf{k},\mathbf{g}-\mathbf{k}z),\\ \left[\frac{d}{dz}+\alpha+\beta_1\,(\mathbf{g}-\mathbf{k}z)^2\right]\Phi_{p1}^{s}(z,\mathbf{k},\mathbf{g}) &= \left[bf-\beta_2\,(\mathbf{g}-\mathbf{k}z)^2\right]\Phi_{p2}^{s}(z,\mathbf{k},\mathbf{g}).\end{aligned}\right\} \tag{43}$$

The solutions to Eqns. (43) can be found with the help of the following two (forward, G_+, and backward, G_-) Green functions (Vladimirov, 1971):

$$G_\pm(z,\mathbf{k},\mathbf{g}) = H(\pm z)\exp\left[\mp\alpha z\mp\beta_1\int_0^z(\mathbf{g}-\mathbf{k}\eta)^2\,d\eta\right], \tag{44}$$

which are solutions to the equations:

$$\left[\pm\frac{d}{dz}+\alpha+\beta_1(\mathbf{g}-\mathbf{k}z)^2\right]G_\pm(z,\mathbf{k},\mathbf{g}) = \delta(z), \tag{45}$$

where $H(z)$ ($H=1$, for $z>0$, $H=0$, for $z\le 0$) is a Heavyside or step function (Morse and Feshbach, 1953).

The final solutions to Eqns. (43) are:

$$\Phi_{p2}^{s}(z,\mathbf{k},\mathbf{g}) = \int G_-\left(z-\xi,\mathbf{k},\mathbf{g}\right)\left[bf-\beta_2(\mathbf{g}-\mathbf{k}\xi)^2\right]F_{p1}^{Q}\left(\xi,\mathbf{k},\mathbf{g}-\mathbf{k}\xi\right)d\xi, \tag{46}$$

$$\Phi^s_{p1}(z,\mathbf{k},\mathbf{g}) = \int G_+\left(z-\xi,\mathbf{k},\mathbf{g}\right)\left[bf-\beta_2(\mathbf{g}-\mathbf{k}\xi)^2\right]\Phi^s_{p2}(\xi,\mathbf{k},\mathbf{g})\Big)\,d\xi + C_1(\mathbf{k},\mathbf{g})\exp\left[-\alpha z-\beta_1\int_0^z(\mathbf{g}-\mathbf{k}\eta)^2\,d\eta\right], \quad (47)$$

where $C_1(\mathbf{k},\mathbf{g})$ is an arbitrary function that is determined by the boundary condition.

After simplification, we obtain the following expressions for F_{pi}:

$$\begin{aligned} F_{p1}(z,\mathbf{k},\mathbf{q}) = F_p^0(\mathbf{k},\mathbf{q}+\mathbf{k}z)\exp[-\alpha z-\beta_1\int_0^z(\mathbf{q}+\mathbf{k}\eta)^2\,d\eta]\Big\{1+\int_0^z d\xi\left[bf-\beta_2(\mathbf{q}-\mathbf{k}\xi)^2\right]\times \\ \times\exp[-\beta_1\int_{\xi-z}^{z}(\mathbf{q}+\mathbf{k}\eta)^2 d\eta]\int_{-\infty}^{0} d\varsigma\left[bf-\beta_2(\mathbf{q}-\mathbf{k}\xi+\mathbf{k}\varsigma)^2\right]\exp[2\alpha\varsigma-\beta_1\int_{\varsigma}^{\xi-\varsigma}(\mathbf{q}-\mathbf{k}\eta)^2 d\eta]\Big\}, \end{aligned} \quad (48)$$

$$F_{p2}(z,\mathbf{k},\mathbf{q}) = F_p^0(\mathbf{k},\mathbf{q}+\mathbf{k}z)\,e^{-\alpha_p z}\int_{-\infty}^{0} d\xi\left[bf-\beta_2(\mathbf{q}-\mathbf{k}z+\mathbf{k}\xi)^2\right]\exp[2\alpha\xi-\beta_1\int_{\xi}^{z-\xi}(\mathbf{q}-\mathbf{k}\eta)^2 d\eta]\ . \quad (49)$$

2.3 Detector Response to the Infinitely Short Laser Pulse Reflected from Water

After the integration of the received radiances over the sensitive area of the detector, the relative (normalized by the pulse power P_0) response of the detector placed at the depth *z* will be

$$\eta_1(z,w) = \frac{\delta(w)\,v\,e^{-\alpha(w+z)}}{1+\left(\frac{\rho_0}{\rho_A}\right)^2\left[1+\frac{4D_1\sigma_0}{3s_0^2}\right]\frac{s_0 z}{\rho_A}} + \int_0^z \frac{\theta(w)(\beta\rho_A)^2\,v\,e^{-\alpha(w+z)}\,d\xi}{2a_1+2D_2\left(2\xi^2-w\xi\right)+D_1\sigma_0\left(\frac{4\xi^3}{3}+2w\xi^2-w^2\xi\right)}, \quad (50)$$

where

$$a_1 = \rho_A^2+\rho_0^2+s_0^2 z^2+D_2 w^2+\frac{D_1\sigma_0}{3}(4z^3+w^3),\quad w = vt-z, \quad (51)$$

$$\left.\begin{aligned} \eta_2(z,w) &= \frac{\beta v}{2D_w}\theta(w)\exp[-\alpha(w+z)], \\ D_w &= 1+\left(\frac{\rho_0}{\rho_A}\right)^2+D_2\left(1+\frac{D_1\sigma_0 w}{3D_2}\right)\left(\frac{w}{\rho_A}\right)^2+\left(1+\frac{4D_1\sigma_0}{3s_0^2}z\right)\left(\frac{s_0 z}{\rho_A}\right). \end{aligned}\right\} \quad (52)$$

At $z=0$, the detector response to the reflected pulse will be:

$$\eta_2(\tau) = \frac{a_0\,\theta(\tau)\,e^{-\tau}}{1+a_2\tau^2+a_3\tau^3}, \quad (53)$$

where

$$\tau = t/t_0\ ,\ t_0 = (\alpha v)^{-1}\ ,\ a_0 = \frac{8\pi^7\omega_0 f}{\left[1-\omega_0(1-f)\right]\left[1+\left(\rho_E/\rho_D\right)^2\right]} \quad (54)$$

$$a_2 = \frac{\pi D_2}{8c^2\left(\rho_E^2+\rho_D^2\right)\left[1-\omega_0(1-f)\right]^2},\quad a_3 = \frac{\pi\omega_0(1-f)D_1}{24c^2\left(\rho_E^2+\rho_D^2\right)\left[1-\omega_0(1-f)\right]^3}, \quad (55)$$

where *f* is the weight coefficient in Eqn. (10).

The values of η_2 calculated according to Eqns. (53)-(55) were compared with the results computed with the Monte Carlo code by Kattawar (1992). The discrepancies between the natural logarithms of the η_2 computed by both these methods for inherent optical properties and phase functions by Petzold (1972) do not exceed 15%.

3.0 CONCLUSIONS

Relatively simple analytical equations derived in a small-angle scattering approximation have been obtained for the infinitely short light pulse reflected by seawater. These equations depend on the inherent optical properties of seawater, as well as the parameters of the pulse and receiver. They can be transformed into equations for an arbitrarily-shaped pulse by a simple convolution procedure. The logarithmic precision of these equations is estimated to be in the range of 15%. The results of this paper can be used for algorithms connected with the rapid assessment of water properties from remote airborne and shipborne platforms.

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