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ALGORITHM FOR COMPUTING APPARENT OPTICAL PROPERTIES OF SHALLOW WATERS UNDER ARBITRARY SURFACE ILLUMINATION*

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ABSTRACT

A self-consistent variant of the two-flow approximation, which takes into account the strong anisotropy of light scattering in a turbid medium, is presented. In order to achieve the precision, comparable with the accuracy of *in situ* measurements, this approach utilizes experimental dependencies between the total mean cosine and the upward mean cosine. The algorithm calculates irradiances, diffuse attenuation coefficients and diffuse reflectances in seawater characterized by arbitrary scattering and absorption coefficients, arbitrary conditions of illumination and a bottom with Lambertian albedo. This theory can be successfully used for calculation of apparent optical properties in both open and coastal oceanic waters, lakes and rivers.

1.0 INTRODUCTION

The majority of existing analytical methods for calculating light fields in scattering media are based on one or another variant of the two-flow approximation to the theory of radiation transfer. The simplicity and convenience of the final results obtained with these approximations and, especially, the possibility of using them to solve inverse problems favorably distinguish the two-flow theories from the more accurate, but much less convenient numerical methods for solving transport equations. In this work we develop a variant of the two-flow approximation that takes into account the strong anisotropy of scattering and the asymmetry of the brightness field at large depths in media such as seawater. The accuracy of the derived formulas in many cases approaches that of the numerical calculations of transport phenomena.

For many practical applications of remote sensing such as the inference of the diffuse attenuation coefficient and component inversion it is sufficient to know only the integral characteristics of the light field such as upward and downward irradiances or reflectances. Present models used in remote sensing applications for radiative transfer employ simple two-flow or quasi-single scattering approximations which suffer from limited validity over the dynamic range of optical properties found in the ocean. However the limitation to open ocean water types restricts the general usage of these models. Remote sensing applications would be greatly enhanced if we add to it a simple model that is applicable to all water types. We present here a self-consistent model that incorporates laboratory and *in situ* measurements of optical properties to encompass the entire range of natural waters.

2.0 APPROACH

We start from the exact equation for irradiances derived from the scalar transfer equation. To make this equation solvable it is necessary, however, to approximate the resulting coefficients in the system of two-flow equations. Due to the inaccuracy inherent in these approximations, the previous approaches as shown in Haltrin (1985) have resulted in insufficient accuracy over some portions of the natural range of optical parameters. We use two main steps to reduce the exact, but analytically unsolvable, system of equations to an approximate system which can be easily solved. The first step consists of replacing the initial arbitrary phase function with the transport phase function. Such substitution of the phase function greatly simplifies the equations, but introduces an excessive error. We reclaim the lost accuracy, in the next step, by introducing empirical relationships between the downward cosine and total mean cosine derived from laboratory and *in situ* data by Timofeyeva (1979). This relationship shows that with the change of Gordon's parameter $x = b_B/(a+b_B) \equiv B\omega_0/(1-\omega_0+B\omega_0)$ from 0 to 1 - here a is the

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absorption coefficient, b_B is the backscattering coefficient, ω_0 is the single scattering albedo and B is the probability of backscattering. The total mean cosine $\bar{\mu}$ also varies from 0 to 1, and the downward mean cosine $\bar{\mu}_d$ decreases from 1 to 0.5.

The main purpose of this work is to obtain equations that relate inherent optical to apparent optical properties for any input radiance distribution. These equations, which are convenient and precise, are valid in the complete range of variability of optical properties of natural water.

In the theory of radiative transfer, the requirements for simplicity and precision are mutually exclusive. Therefore, for a successful resolution of the problem, we have accepted a compromise by determining the degree of simplicity and precision. We will refer to our method as the *self-consistent method*. The idea of this method was adopted from classical mechanics (Gantmakher, 1970). For a better understanding of this idea we quote here the following example . Suppose we have to obtain the equation of motion of a material body around some center of attraction. The law of attraction is unknown to us, or it is known only partially, but in addition we have some information about the shape of the trajectories in the form of dependencies between integral parameters of these trajectories. This problem can be solved provided we use the available information to constrain the acceptable solutions. In this example the knowledge of additional information on *consequences* (trajectory parameters) has made it possible to compensate for the lack of information on *causes* (attraction forces).

In the theory of radiative transfer the main *causes* are inherent optical properties such as the absorption coefficient and volume scattering function. The main *consequences* are apparent optical properties, such as the angular distribution of radiance, as a function of depth. In general, the volume scattering function is only approximately known, with an unknown precision. It is impossible in general to calculate the volume scattering function of an actual medium because in many cases the shape of the scattering particles is irregular and often *exotic*. In addition, the optical characteristics of these particles are known only approximately. Experimental measurement of the volume scattering function in the small-angles range becomes complicated due to difficulty in discriminating between unscattered and forward scattered light. The measurement of the volume scattering function in the range of angles close to the backward direction is impossible in principle. This is due to the fact that one cannot install a receiver before or behind an emitter without considerable distortion in the process of measurement. To overcome this, a beam splitting of backscattered light has been utilized with some success.

At the same time, the angular distribution of the scattered light at depth is always less anisotropic than the volume scattering function. In addition, the anisotropy of the direct light of the outer sources is known. For these reasons measurement of radiance distribution is less difficult, and the precision of these measurements is restricted only by the perfection of the measuring device.

Thus, in attempting to solve the problem of light field calculation in a scattering and absorbing medium, we restrict ourselves to the simplest transport approximation to the volume scattering function. The information, which we lose through this simplification, is restored afterwards by accepting functional dependencies between integral parameters of the radiance angular distribution, which are derived from an approximation to experimental data.

Now let us start from the scalar equation describing the transport of optical radiation in a sea with depth z_B:

$$\cos\theta \frac{\partial \tilde{L}(z,\theta,\varphi)}{\partial z} + c\,\tilde{L}(z,\theta,\varphi) = \frac{b}{4\pi}\int \tilde{L}(z,\theta',\varphi')\,p(\gamma)\,d\Omega' \,. \tag{1}$$

The system of coordinates is chosen in such a way that the xy-plane coincides with the sea surface on which the radiation is incident, while the $0z$-axis is oriented into the medium. Here $\tilde{L}(z,\theta,\varphi)$ is the total radiance of the light, and θ and φ are the zenith and azimuthal angles which determine the direction of the light propagation, measured from the positive direction of the $0z$-axis; $c = a + b$ is the beam attenuation coefficient, a is the absorption coefficient, b is the scattering coefficient, $d\Omega' = \sin\theta'\,d\theta'\,d\varphi'$ is the element of solid angle; $p(\gamma)$ is the light scattering phase-function, where γ is the light-scattering angle which is determined by the relation

$$\cos\gamma = \mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\,\cos(\varphi-\varphi')\,, \tag{2}$$

where $\mu = \cos\theta$, $\mu' = \cos\theta'$, and the scattering phase-function is normalized as follows:

$$\int_{4\pi} p(\gamma)\,d\Omega' = 4\pi\,. \tag{3}$$

In anisotropically light-scattering seawater, the scattering phase-function $p(\gamma)$ has a distinct diffraction peak near the forward direction $\gamma = 0$ (Haltrin, 1997). The light rays, scattered in a small solid angle near $(\gamma \approx 0)$ form the halo part of the scattered light. They are, for all practical purposes, indistinguishable from the unscattered light.

This suggests that the halo part of the rays should be regarded as unscattered light. Mathematically it means that the diffraction peak in the forward direction should be removed from the scattering function.

We shall separate the main part of the halo rays by representing the scattering phase function as a sum of isotropic and anisotropic parts:

$$p(\gamma) = 2B + (1-2B)\,p_\delta(\gamma)\,, \qquad p_\delta(\gamma) = \frac{p(\gamma)-2B}{1-2B}, \quad \int_{4\pi} p_\delta(\gamma)\,d\Omega' = 4\pi\,, \qquad B = 0.5\int_{\pi/2}^{\pi} p(\gamma)\sin\gamma\,d\gamma\,, \quad (4)$$

where $\delta(x)$ is the Dirac delta function (Morse, and Feshbach, 1953) and B is the probability of backscattering. The following relation is valid when the elongation of the scattering phase function is increased (Potter, 1970):

$$\lim_{B\to 0} p_\delta(\gamma) = 2\,\delta(1-\cos\gamma) \equiv 4\pi\delta(\varphi-\varphi')\,\delta(\mu-\mu'). \quad (5)$$

Next, let us introduce the following auxiliary scattering phase function:

$$\tilde{p}(\gamma) = 2B + 2\,(1-2B)\,\delta(1-\cos\gamma), \quad \int_{4\pi} \tilde{p}(\gamma)\,d\Omega' = 4\pi\ . \quad (6)$$

By replacing $p(\gamma)$ with its equivalent $\tilde{p}(\gamma)+[p(\gamma)-\tilde{p}(\gamma)]$, we rewrite the radiative transfer equation (1) in the following adequate form:

$$\left(\mu\frac{\partial}{\partial z}+\alpha\right)\tilde{L}(z,\mu,\varphi) = \frac{b_B}{2\pi}\int \tilde{L}(z,\mu',\varphi')\,d\Omega' + \frac{b}{4\pi}\int\left[p(\gamma)-\tilde{p}(\gamma)\right]\tilde{L}(z,\mu',\varphi')\,d\Omega', \quad (7)$$

where $\alpha = a + 2\,b_B$ is the renormalized extinction coefficient and $b_B = bB$ is the backscattering coefficient.

Let $L_q(\mu,\varphi)$ be the radiance of light just below the surface, *i.e.* at $z = +0$, and let $L(z,\mu,\varphi)$ be the radiance of the scattered component minus the forward scattered rays. In this case the total radiance at the depth z, $\tilde{L}(z,\mu,\varphi)$, is represented as a sum of two components: 1) the rays scattered in all directions minus the forward scattered light, $L(z,\mu,\varphi)$, and 2) the radiance which is a sum of the direct and forward scattered light:

$$\tilde{L}(z,\mu,\varphi) = L(z,\mu,\varphi) + \begin{cases} L_q(\mu,\varphi)\exp(\alpha z/\mu), & 0<\mu\le 1, \\ 0\,, & -1\le\mu\le 0\,. \end{cases} \quad (8)$$

The coefficient α also can be regarded as the beam attenuation coefficient for the sum of the unscattered and forward scattered light. In the right part of Eqn. (8) we did not include the light reflected from the bottom. By doing this, we assume that either the sea is optically thick $(\alpha z_B >> 1)$ or its bottom reflects light according to Lambert's law. Substitution of Eqn. (8) into the exact equation (7) gives us the following equation for the radiance of the scattered light:

$$\left(\mu\frac{\partial}{\partial z}+\alpha\right)L(z,\mu,\varphi) = \frac{b_B\,E^0(z)+g(z,\mu,\varphi)+\Delta(z,\mu,\varphi)}{2\pi}, \quad (9)$$

where $E^0(z) = \int_0^{2\pi} d\varphi' \int_{-1}^{1} L(z,\mu',\varphi')\,d\mu'$ is the scalar irradiance due to diffuse light, and $g(z,\mu,\varphi)$ is the source function:

$$g(z,\mu,\varphi) = \frac{b}{2}\int_0^{2\pi} d\varphi' \int_{-1}^{1} p(\gamma)\,L_q(\mu',\varphi')\,e^{-\alpha z/\mu'}\,d\mu' - 2\pi\,b\,(1-2\,B)\,L_q(\mu,\varphi)\,e^{-\alpha z/\mu}, \quad (10)$$

$$\Delta(z,\mu,\varphi) = \frac{b}{2}\int_0^{2\pi} d\varphi' \int_{-1}^{1}\left[p(\gamma)-\tilde{p}(\gamma)\right]L(z,\mu',\varphi')\,d\mu'. \quad (11)$$

The function Δ, given by Eqn. (11), describes all the differences in scattering between the real seawater model with phase function $p(\gamma)$ and a simple analytically solvable model with the transport scattering function $\tilde{p}(\gamma)$. Equation (9) is completely equivalent to Eqn. (1). The approximation given by Eqn. (9) corresponds to the inclusion of the halo rays in the non-scattered light radiance in the forward direction.

The value of the function Δ, given by Eqn. (11), vanishes for two limiting cases: 1) for isotropic scattering when $p(\gamma)=1$, and $B = 0.5$, and 2) for extremely anisotropic scattering when $p(\gamma) = 2\,\delta(1-\cos\gamma)$, and $B = 0$.

3.0. EQUATIONS FOR IRRADIANCES

Equation (9) for an arbitrary phase function $p(\gamma)$ cannot be solved analytically. By neglecting the term Δ compared to the term $b_B E^0 + g$, the problem is reduced to the case of isotropic scattering. This case can be solved exactly at any depth in the scattering medium. However, the ellipsoidally shaped radiance distribution, obtained as a solution in this case, very poorly describes the experimental results by Timofeyeva (1979).

Let us seek the solution for the problem given by Eqn. (9) in the two-flow approximation. We do this by formally setting the function $\Delta = 0$ and assuming that the radiance distribution of scattered light within the ocean is described by the following asymptotic formula

$$L(\mu,z) \propto L^{\infty}(\mu)\, \exp(-\kappa\, c\, z), \qquad L^{\infty}(\mu) = (1-\bar{\mu}^2)^2 \big/ (1-\bar{\mu}\,\mu)^3, \qquad \bar{\mu} = \int_0^1 L^{\infty}(\mu)\,\mu\, d\mu \Big/ \int_0^1 L^{\infty}(\mu)\, d\mu. \quad (12)$$

The angular radiance distribution $L^{\infty}(\mu)$ in Eqn. (12) is derived from experimental results by Timofeyeva (1979). Here κ is the parameter for the deep regime, $\bar{\mu}$ is the total mean cosine, and $L^{\infty}(\mu)$ is the radiance of light at the depths where the asymptotic light regime is established (Haltrin, 1985).

To derive two-flow equations we first introduce irradiances E_i and scalar irradiances E_i^0 from above $(i=1)$ and from below $(i=2)$ by the following formulas:

$$E_1(z) = \int_0^{2\pi} d\varphi \int_0^1 L(z,\mu,\varphi)\,\mu\, d\mu, \quad E_2(z) = -\int_0^{2\pi} d\varphi \int_{-1}^0 L(z,\mu,\varphi)\,\mu\, d\mu, \quad (13)$$

$$E_1^0(z) = \int_0^{2\pi} d\varphi \int_0^1 L(z,\mu,\varphi)\, d\mu, \quad E_2^0(z) = \int_0^{2\pi} d\varphi \int_{-1}^0 L(z,\mu,\varphi)\, d\mu. \quad (14)$$

Here the index $i=1$ corresponds to downwelling irradiances and the index $i=2$ corresponds to upwelling irradiances, *i. e.* $E_1 \equiv E_d$, $E_2 \equiv E_u$, $E_1^0 \equiv E_d^0$, $E_2^0 \equiv E_u^0$. The average downward $\mu_1(z) \equiv \mu_d(z)$ and upward $\mu_2(z) \equiv \mu_u(z)$ cosines are equal to

$$\mu_d(z) \equiv \mu_1(z) = E_1(z)/E_1^0(z), \quad \mu_u(z) \equiv \mu_2(z) = E_2(z)/E_2^0(z). \quad (15)$$

By substituting into Eqn. (15) the depth radiance in the sea given by Eqn. (12), the following formulas for downward and upward cosines are obtained:

$$\mu_1(z) \to \bar{\mu}_1 = 1/(2-\bar{\mu}), \qquad \mu_2(z) \to \bar{\mu}_2 = 1/(2+\bar{\mu}). \quad (16)$$

According to the experiments and *in situ* measurements performed by Timofeyeva (1979), the relation $\bar{\mu}_1 = (2-\bar{\mu})^{-1}$ is valid with high accuracy inside modeled scattering and absorbing media and in the sea where the asymptotic or depth regime is established.

Using Eqn. (9) let us derive a system of two-flow equations for irradiances. First, let $\Delta = 0$. Then, let us apply to Eqn. (9) the integral operators:

$$\int_o^{2\pi} d\varphi \int_0^1 \mu\, d\mu \ldots, \qquad \int_o^{2\pi} d\varphi \int_{-1}^0 \mu\, d\mu \ldots. \quad (17)$$

By using Eqns. (13) -(15), and replacing the average cosines $\mu_i(z)$ by their asymptotic values given by Eqns. (16), the following matrix equation for downward and upward irradiances E_1 and E_2 is obtained:

$$\hat{T}_{ik}\, E_k(z) = f_i(z), \quad (18)$$

where the differential matrix operator $\hat{T}_{ik}$ has the following form:

$$\hat{T}_{ik} = \begin{Vmatrix} \dfrac{d}{dz} + (2-\bar{\mu})(a+b_B) & -(2+\bar{\mu})\, b_B \\ -(2-\bar{\mu})\, b_B & -\dfrac{d}{dz} + (2+\bar{\mu})(a+b_B) \end{Vmatrix}. \quad (19)$$

The source functions f_1 and f_2 on the right side of Eqn. (18) can be expressed through the radiance distribution just below the sea surface and the scattering phase function $p(\gamma)$:

$$f_1(z) = b \int_0^{2\pi} d\varphi \int_0^1 d\mu \left[2B - \psi(\mu)\right] L_q(\mu,\varphi)\, e^{-\alpha z/\mu}, \quad (20)$$

$$f_2(z) = b\int_0^{2\pi} d\varphi \int_0^1 d\mu\, \psi(\mu)\, L_q(\mu,\varphi)\, e^{-\alpha z/\mu}\,, \tag{21}$$

where

$$\psi(\mu) = \frac{1}{2}\int_0^1 \bar{p}(-\mu',\mu)\, d\mu'\,, \quad \bar{p}(\mu,\mu') \equiv \int_0^{2\pi} p(\gamma)\frac{d\varphi}{2\pi}\,. \tag{22}$$

In Eqn. (22) and further in this paper, the repeated indices imply summation: $g_i h_i \equiv \sum_i g_i h_i$.

The negative eigenvalue of the system of equations (16) is given by the equation:

$$-\alpha_\infty = \bar{\mu}\,(a + b_B) - \sqrt{4a\,(a + 2\,b_B) + \bar{\mu}^2 b_B^2}\,. \tag{23}$$

On the other hand, the exact eigenvalue of both Eqns. (1) and (9) is determined by Gershun's formula:

$$\alpha_\infty = \kappa\, c \equiv a/\bar{\mu}\,. \tag{24}$$

Now we impose the condition that the attenuation coefficients given by Eqns. (23) and (24) match. In other words, we force our radiance distribution to match the experimental one given by Eqn. (12). In this case, we can find α_∞ and $\bar{\mu}$ as functions of the inherent optical parameters of the medium a and b_B. Solving Eqns. (23) and (24) relative to $\bar{\mu}$, we have the following formula:

$$\bar{\mu} = a \quad \sqrt{a + 3\,b_B + \sqrt{b_B\,(4a + 9\,b_B)}}\,. \tag{25}$$

From Eqns. (24) and (25) we obtain the following equation for the deep regime parameter (or ratio of the diffuse attenuation coefficient to the beam attenuation coefficient c):

$$\kappa = \sqrt{(1-\omega_0)\left\{1 - \omega_0 + 3\,B\omega_0 + \sqrt{B\omega_0\left[4(1-\omega_0) + 9\,B\omega_0\right]}\right\}}\,, \tag{26}$$

where $\omega_0 = b/c \equiv b/(a+b)$ is the single-scattering albedo.

4.0. SOLUTIONS FOR IRRADIANCES

Let us seek the solution of Eqn. (18) as the sum of the general and the partial solutions (Morse and Feshbach, 1953):

$$E_i(z) = A\,a_i\, \exp(-\alpha_\infty z) + E\,e_i\, \exp(\alpha_0 z) + \int_0^{z_B} G_{ik}(z - z')\, f_k(z')\; dz' \tag{27}$$

where the value

$$\alpha_0 = \bar{\mu}\,(a + b_B) + \sqrt{4a\,(a + 2\,b_B) + \bar{\mu}^2 b_B^2} \tag{28}$$

is the second eigenvalue of Eqns. (18), and $a_1 = e_2 = 1$, $a_2 = R_\infty$, $e_1 = R_0$, where the parameters,

$$R_\infty = \frac{(2-\bar{\mu})\,b_B}{(2+\bar{\mu})(a+b_B) + \alpha_\infty} = \frac{(2-\bar{\mu})(a+b_B) - \alpha_\infty}{(2-\bar{\mu})\,b_B} = \frac{2-\bar{\mu}}{2+\bar{\mu}}\,R_0\,, \tag{29}$$

$$R_0 = \frac{(2+\bar{\mu})\,b_B}{(2-\bar{\mu})(a+b_B) + \alpha_0} = \frac{(2+\bar{\mu})(a+b_B) - \alpha_0}{(2-\bar{\mu})\,b_B} = \frac{2+\bar{\mu}}{2-\bar{\mu}}\,R_\infty\,, \tag{30}$$

are the diffuse reflection and the spherical diffuse reflection coefficients, respectively. The constants A and E in Eqn. (27) are determined by the boundary conditions. The matrix $G_{ik}(z)$ is the Green's matrix. It satisfies the following fundamental matrix equation (Morse and Feshbach, 1953):

$$\hat{T}_{il}\, G_{lk}(z) = \delta_{ik}\, \delta(z)\,, \tag{31}$$

where δ_{ik} is the Kronecker's symbol (or the unity matrix). After some algebra it is possible to show that the Green's matrix has the following form:

$$G_{ik}(z) = \begin{Vmatrix} 1 & R_0 \\ R_\infty & R_0 R_\infty \end{Vmatrix} \frac{H(z)\, e^{-\alpha_\infty z}}{1 - R_0 R_\infty} + \begin{Vmatrix} R_0 R_\infty & R_0 \\ R_\infty & 1 \end{Vmatrix} \frac{H(-z)\, e^{\alpha_0 z}}{1 - R_0 R_\infty}\,, \tag{32}$$

where $H(z)$ is the Heaviside or step function:

$$H(z) = \begin{cases} 1, & z > 0, \\ 0, & z \le 0. \end{cases} \qquad (33)$$

First, let us apply Eqn. (32) to Eqn. (27). Then, we impose the following two boundary conditions at the depths corresponding to the surface $(z = 0)$ and the bottom $(z = z_B)$

$$E_1(0) = E_0, \quad E_2(z_B) = A_B\left[E_1(z_B) + E_1^f(z_B)\right], \qquad (34)$$

where A_B is the albedo of the bottom, and E_1^f is the total downward irradiance created by the direct and forward scattered light. It is given by the equation:

$$E_1^f(z) = \int_0^{2\pi} d\varphi \int_0^1 L_q(\mu,\varphi)\, \exp(-\alpha z/\mu)\, \mu\, d\mu, \qquad E_2^f(z) = 0. \qquad (35)$$

These calculations result in the following two equations for the descending and ascending irradiances of diffuse light:

$$E_1(z) = \left[E_0 + M(z)\right] e^{-\alpha_\infty z} + R_0\, N(z)\left(e^{\alpha_0 z} - e^{-\alpha_\infty z}\right), \qquad (36)$$

$$E_2(z) = R_\infty\left[E_0 + M(z)\right] e^{-\alpha_\infty z} + N(z)\left(e^{\alpha_0 z} - R_0 R_\infty e^{-\alpha_\infty z}\right), \qquad (37)$$

where the auxiliary functions are given by the equations:

$$M(z) = \frac{1}{1 - R_0 R_\infty} \int_0^z dz' \{[f_1(z') + R_0 f_2(z')] e^{\alpha_\infty z'} - R_0 [R_\infty f_1(z') + f_2(z')] e^{-\alpha_0 z'}\}, \qquad (38)$$

$$N(z) = \frac{A_B - R_\infty}{R_0 \Delta_B}\left[E_0 + M(z_B)\right] e^{-\nu z_B} + \frac{1}{1 - R_0 R_\infty} \int_0^{z_B} dz' [R_\infty f_1(z') + f_2(z')] e^{-\alpha_0 z'} + + \frac{A_B}{R_0 \Delta_B} \int_0^{2\pi} d\varphi \int_0^1 L_q(\mu,\varphi)\, e^{-(\alpha_0 + \alpha/\mu) z_B}\, \mu\, d\mu, \qquad (39)$$

$$\Delta_B = \left(1/R_0 - A_B\right) + \left(A_B - R_\infty\right)\exp(-\nu z_B), \qquad \nu = \alpha_0 + \alpha_\infty. \qquad (40)$$

For the case of totally diffuse illumination of the sea surface we make two assumptions. First, the irradiance of light from the external sources, which is transmitted through the upper boundary, is given by equation:

$$E_q^e = E_1^f(0) \int_0^{2\pi} d\varphi \int_0^1 L_q(\mu,\varphi)\, \mu\, d\mu. \qquad (41)$$

Second, this light is completely diffuse. In the next step, we take this into account by applying the boundary condition $E_1(0) = E_q^0$ to the solution and setting the external sources $f_i(z)$ to be equal to zero. Based on these steps, the following substitutions are made:

$$E_0 = E_q^0, \quad M(z) = 0, \quad N(z) = \left[E_q^0\left(A_B - R_\infty\right) / \left(R_0\, \Delta_B\right)\right]\exp(-\nu z_B). \qquad (42)$$

In this particular case Eqns. (36) and (37) are be transferred into the following solutions for the downward and upward irradiances:

$$E_d(z) \equiv E_1(z) = \left(E_q^0 / \Delta_B\right)\left\{\left(1/R_0 - A_B\right)\exp(-\alpha_\infty z) + (A_B - R_\infty)\exp[-\alpha_0(z_B - z) - \alpha_\infty z_B]\right\} \qquad (43)$$

$$E_u(z) \equiv E_2(z) = \left(E_q^0 / \Delta_B\right)\left\{R_\infty\left(1/R_0 - A_B\right)\exp(-\alpha_\infty z) + \left[(A_B - R_\infty)/R_0\right]\exp[-\alpha_0(z_B - z) - \alpha_\infty z_B]\right\}. \qquad (44)$$

Equations (43) and (44) can be used to calculate diffuse attenuation, reflection and transmission coefficients of shallow waters with arbitrary inherent optical properties a, b and b_B, arbitrary depth and bottom albedo A_B.

5.0. TRANSMISSION AND DIFFUSE REFLECTION COEFFICIENTS

Let us now calculate the transmission coefficient of the seawater layer $(0,z)$ for the diffuse light $T(z)=E_1(z)/E_1(0)$, and the diffuse reflectance coefficient (DRC) $R(z)=E_2(z)/E_1(z)$. Using Eqns. (43)-(44), we obtain the following formulas for the transmission and diffuse reflectance coefficients:

$$T(z)=\frac{\left(1-R_0\,A_B\right)+R_0\,(A_B-R_\infty)\,\exp[-\nu(z_B-z)]}{\left(1-R_0\,A_B\right)+R_0\,(A_B-R_\infty)\exp(-\nu\,z_B)}\exp(-\alpha_\infty z), \tag{45}$$

$$R(z)=\frac{R_\infty\left(1-R_0\,A_B\right)+(A_B-R_\infty)\exp[-\nu(z_B-z)]}{\left(1-R_0\,A_B\right)+R_0\,(A_B-R_\infty)\exp[-\nu(z_B-z)]}, \tag{46}$$

where $\nu=\alpha_0+\alpha_\infty\equiv 2a\left(\bar{\mu}/(1-x)+1/\bar{\mu}\right)$. The diffuse reflection of an ocean with depth z_B and Lambertian bottom albedo A_B is obtained by setting $z=0$ in Eqn. (46):

$$R=\frac{R_\infty\left(1-R_0\,A_B\right)+(A_B-R_\infty)\exp(-\nu z_B)}{\left(1-R_0\,A_B\right)+R_0(A_B-R_\infty)\exp(-\nu z_B)}. \tag{47}$$

Equation (47) generalizes the Kubelka-Munk (1931) formula for the depth dependent diffuse reflectance coefficient to the equation that is applicable to seawater of arbitrary turbidity.

Setting the limit $\nu\,z_B\to\infty$ in Eqn. (47) gives us the diffuse reflectance of an optically deep sea R_∞: $R\to R_\infty$. After appropriate calculations we obtain the following working equations:

$$R_\infty=\left(\frac{1-\bar{\mu}}{1+\bar{\mu}}\right)^2,\quad \bar{\mu}=\sqrt{\frac{1+2x-\sqrt{x\,(4+5x)}}{1+x}},\quad x=\frac{b_B}{a+b_B}\equiv\frac{B\omega_0}{1-\omega_0+B\omega_0}. \tag{48}$$

Equations (48) are valid for any arbitrary values of a and b_B. They can be used for calculations of diffuse reflectance coefficient of an optically thick ocean with arbitrary turbidity.

6.0. ASYMPTOTICS

Let us investigate asymptotics of Eqns. (48) for the cases of small and large values of the ratio b_B/a. In the limit $b_B/a\to 0$, when absorption predominates, we have:

$$R_\infty=\frac{b_B}{4a}+\frac{1}{4}\left(\frac{b_B}{a}\right)^{3/2}-\frac{3}{32}\left(\frac{b_B}{a}\right)^{5/2}+\frac{5}{64}\left(\frac{b_B}{a}\right)^3,\quad \frac{b_B}{a}<<1. \tag{49}$$

For media with low absorption the following equation is valid:

$$R_\infty=\left(1-2\sqrt{\frac{a}{6\,b_B}}\right)\Big/\left(1+2\sqrt{\frac{a}{6\,b_B}}\right),\quad \frac{b_B}{a}>>1. \tag{50}$$

In the case of isotropic scattering, Eqn. (50) coincides with the asymptotically correct formula derived by Gate (1974).

8.0. INVERSE PROBLEM

To solve the inverse problem of calculating inherent optical properties of the ocean *via* remotely measured diffuse reflection coefficient, it is necessary to have a relation that expresses the absorption coefficient a in terms of the experimentally measurable diffuse reflectance coefficient R_∞ and the backscattering coefficient b_B. Using Eqns. (48)-(49), we obtain

$$a=b_B\,\Phi_H(R_\infty),\quad \Phi_H(R_\infty)=\left(1-\sqrt{R_\infty}\right)^2\left(1+4\sqrt{R_\infty}+R_\infty\right)\Big/\left(4\,R_\infty\right). \tag{51}$$

Comparison with Monte Carlo calculations shows that for media with $x<0.5$ the function $\Phi_H(R_\infty)$ is much more precise than the function $\Phi_K=\left(1-R_\infty\right)^2\big/\left(2R_\infty\right)$ derived by Kubelka-Munk (1931), especially for smaller diffuse reflectances R_∞ typical of open and coastal ocean waters.

7.0. ARBITRARY ILLUMINATION

Let us consider the case of arbitrary illumination of the surface of an optically infinitely thick ocean. In this case the diffuse reflection coefficient is given by the equation:

$$R = (1-\bar{\mu})^2 \int_0^{2\pi} d\varphi \int_0^1 \left[1 + \frac{2\bar{\mu}}{1+\bar{\mu}^2}\,\frac{\psi(\mu)-B}{B}\right] \frac{J(\mu,\varphi)\,\mu\,d\mu}{1+\mu\,\bar{\mu}\,(4-\bar{\mu}^2)}, \qquad (52)$$

where $J(\mu,\varphi) \equiv E_q(\mu,\varphi)/E_q^0$ is the normalized distribution of the light radiance transmitted through the sea surface.

For the direct sun illumination under the zenith and azimuth angles $(\cos^{-1}\mu_s, \varphi_s)$, the normalized radiance is given by the formula: $J_s(\mu,\varphi) \equiv \mu_s^{-1}\delta(\varphi-\varphi_s)\,\delta(\mu-\mu_s)$. Substituting value $J = J_s$ into Eqn. (52), we obtain the equation for the diffuse reflection of the ocean illuminated by the direct sunlight:

$$R_s = \frac{(1-\bar{\mu})^2}{1+\mu_s\,\bar{\mu}\,(4-\bar{\mu}^2)}\left[1+\frac{2\bar{\mu}}{1+\bar{\mu}^2}\,\frac{\psi(\mu_s)-B}{B}\right], \quad \psi(\mu) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{0} p(\gamma)\,d\varphi\,d\mu' . \qquad (53)$$

For the case of the combined illumination of the sea by the direct light of the sun and the diffuse sky light, using Eqns. (52), it is easy to obtain the equation for the diffuse reflection coefficient of an optically deep ocean:

$$R_c = \left(R_\infty + qR_s\right)/\left(1+q\right). \qquad (54)$$

Here R_∞ and R_s are given by Eqns. (48) and (30), respectively, $\mu_s = \sqrt{1-\sin^2 Z_\oplus / n_w^2}$ is the cosine of the angle at which the solar rays enter the sea; $Z_\oplus$ is the sun zenith angle, n_w is the index of refraction of seawater, $q = E_s / E_0$, where E_s is the irradiance by direct sunlight, at $z = +0$, E_0 is the irradiance of a horizontal area at $z = +0$ by the ambient light of the sky.

8.0. CONCLUSIONS

A self-consistent variant of the two-flow approximation, which takes into account the strong anisotropy of light scattering in turbid media, is presented. In order to achieve precision, comparable with the accuracy of *in situ* measurements, this approach utilizes experimental dependencies between mean cosines of light in seawater. It calculates irradiances, diffuse attenuation coefficients and diffuse reflectances in waters with arbitrary scattering and absorption coefficients, arbitrary conditions of illumination and a bottom with Lambertian albedo. This theory can be successfully used for computation of apparent optical properties in open and coastal oceanic waters, lakes and rivers.

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