Optical diffuse reflection of deep and shallow stratified sea waters

Vladimir I. Haltrin †
Naval Research Laboratory, Ocean Sciences Branch, Code 7331
Stennis Space Center, MS 39529-5004, USA

ABSTRACT

Differential equations for the transmittance and the diffuse reflection coefficients of the stratified sea are obtained. To simplify starting radiative transfer equation, the approach uses experimental dependencies between mean cosines of underwater angular radiance distribution. The resulting equation for the diffuse reflection coefficient is of a Riccati type. For the homogeneous sea with arbitrary combination of inherent optical properties, that equation is solved analytically. For vertically inhomogeneous sea it is solved approximately. The resulting formula expresses diffuse reflection coefficient through the profiles of inherent optical properties of seawater and bottom depth and albedo. The results of calculations with main formula are compared with the Monte Carlo computations. It was found that the precision of this theory is about 15% and it is comparable with the precision of contemporary *in situ* measurements.

Keywords: Diffuse reflection coefficient, sea optical properties, radiative transport, inhomogeneous waters, coastal waters.

1. INTRODUCTION

Ocean optics community now makes extensive use of relations that relates the diffuse reflection coefficient of a homogeneous optically infinitely deep ocean through inherent optical properties of seawater:

$$R = k\, b_B \big/ \left(a + b_B\right), \tag{1}$$

and

$$R = k\, b_B / a , \tag{2}$$

here a is the absorption coefficient, b_B is the backscattering coefficient, and k is a numerical coefficient that varies from 0.2 to 0.5, depending on publication. Several versions of Eqns. (1) - (2) were derived with the two-flux approximation of radiation transfer theory [1]. The applicability of the radiative transfer theory (RTT) to the ocean-optics problems has been verified by numerous authors in ground truth measurements and model experiments [2]. So, it is possible to accept, that RTT can be successfully applied in calculations of oceanic light fields.

However, real marine waters are inhomogeneous. Optical properties of natural waters are always vertically stratified. This optical stratification varies depending on the area, the time of year, and hydrologic conditions. Consequently, it is very important to derive an expression for the diffuse reflection coefficient of the ocean suitable for practical use with arbitrary vertical profiles of the optical characteristics.

2. BASIC EQUATIONS

Let us start with the system of the two-flow equations for light irradiances obtained in Refs. [3, 4]:

$$\left.\begin{aligned} \left[\frac{d}{dz} + (2-\bar{\mu})(a+b_B)\right] E_d(z) - (2+\bar{\mu})\, b_B\, E_u(z) &= 0, \\ -(2-\bar{\mu})\, b_B\, E_d(z) + \left[-\frac{d}{dz} + (2+\bar{\mu})(a+b_B)\right] E_u(z) &= 0, \end{aligned}\right\} \tag{3}$$

† For further information: e-mail: <haltrin@nrlssc.navy.mil>, phone: 228-688-4528, fax: 228-688-5379.

here $E_d(z)$ and $E_u(z)$ are, respectively, the downward and upward irradiances, z is the depth coordinate (the Cartesian axis $0z$ is normal to the sea surface and directed to the bottom), $\bar{\mu}$ is the average cosine over the deep water angular distribution of radiance,[5] $b_B = b\,B$, b is the scattering coefficient,

$$B = 0.5\int_{\pi/2}^{\pi} p(\cos\theta)\sin\theta\, d\theta \tag{4}$$

is the backscattering probability, and $p(\cos\theta)$ is the phase function of scattering at the angle θ.

System of Eqns. (3) was derived from the radiation transfer equation under the following assumptions:

1. The scattering phase function was represented in the transport approximation as the sum of isotropic and δ-shaped phase functions: $p(\cos\gamma) = 2B + 2(1-B)\delta(1-\cos\gamma)$ where $\delta(1-\cos\gamma)$ is the Dirac's delta function.

2. The deep water angular radiance distribution, used in calculations of the downward and upward average cosines μ_d and μ_u, was represented by a relation that closely approximates the results observed in Ref. [6]:

$$\left.\begin{aligned} &L_\infty(\mu) = \frac{\left(1-\bar{\mu}^2\right)^2}{\left(1-\bar{\mu}\,\mu\right)^3}, \quad \bar{\mu} = \frac{\int_{-1}^{1} L_\infty(\mu)\,\mu\, d\mu}{\int_{-1}^{1} L_\infty(\mu)\, d\mu}, \quad \mu = \cos\theta, \\ &L_\infty(\mu) = \frac{1-\bar{\mu}^2}{2\bar{\mu}}\sum_{n=0}^{\infty}(2n+1)Q_n^2\left(1/\bar{\mu}\right)P_n(\mu) \equiv \frac{1-\bar{\mu}^2}{2\bar{\mu}}\sum_{n=0}^{\infty}(2n+1)\frac{d^2Q_n(\xi)}{d\xi^2}P_n(\mu)\Big|_{\xi=1/\bar{\mu}}, \end{aligned}\right\} \tag{5}$$

here $Q_n(\xi)$ is a Legendre polynomial of the second kind [7]. The upward and downward mean cosines over radiance distribution (5) are equal to:

$$\left.\begin{aligned} \bar{\mu}_d &= \int_0^1 L_\infty(\mu)\,\mu\, d\mu \bigg/ \int_0^1 L_\infty(\mu)\, d\mu, \quad \bar{\mu}_d = 1/(2-\bar{\mu}), \\ \bar{\mu}_u &= -\int_{-1}^{0} L_\infty(\mu)\,\mu\, d\mu \bigg/ \int_{-1}^{0} L_\infty(\mu)\, d\mu, \quad \bar{\mu}_u = 1/(2+\bar{\mu}). \end{aligned}\right\} \tag{6}$$

3. The average cosine $\bar{\mu}$ was expressed through the inherent optical properties of the medium a and b_B as follows: a) by solving the system of Eqns. (3) in the lower half-space, we obtain the following relation for the diffuse attenuation coefficient in the depth regime:

$$k_\infty = \sqrt{4a(a+2b_B) + \bar{\mu}^2 b_B^2} - \bar{\mu}(a+b_B). \tag{7}$$

b) by jointly solving Eqn. (7) with the exact Gershun's relation, $k_\infty = a/\bar{\mu}$, we find the average cosine $\bar{\mu}$. It is expressed as a function of the absorption and backscattering coefficients, a and b_B, or Gordon's parameter $g = b_B/(a+b_B)$, $(0 \le g \le 1)$:

$$\bar{\mu} = \frac{a}{\sqrt{a + 3b_B + \sqrt{b_B(4a+9b_B)}}} \equiv \sqrt{\frac{1-g}{1+2g+\sqrt{g(4+5g)}}}. \tag{8}$$

The use of both assumptions leads to a very accurate version of the two-flow approximation [8]. This approximation significantly improves the Schwartzschild [9] – Schuster [10] and the Kubelka-Munk [11] two-flow theories, especially for the case of seawater. The formula obtained for the asymptotic diffuse attenuation coefficient with the described theory is:

$$k_\infty = c\sqrt{(1-\omega_0)\left[1-\omega_0(1-3B)\right] + \sqrt{B\omega_0(4-4\omega_0+9B\omega_0)}}, \tag{9}$$

here $c = a + b$ is the beam attenuation coefficient. This equation is valid for all values of backscattering probability B and single scattering albedo $\omega_0 = b/c$. The relative error of Eqn. (9) does not exceed 5%. For the medium with strongly anisotropic scattering $(B << 0.02)$ and arbitrary ω_0 or for the medium with $\omega_0 \geq 0.7$ and arbitrary B, the accuracy of Eqn. (9) approaches the accuracy of numerical methods [4].

Unlike in many other two-flow theories the derivation of system of Eqns. (3) does not imply that the inherent optical properties a and b_B are fixed. Possible existence of a local depth regime, characterized by the angular radiance distribution given by Eqn. (5) with the parameter $\bar{\mu}$ that varies with depth, is proved by the Timofeyeva's experiments [6]. On the basis of this, we may assume that the Eqns. (3) are also valid for the stratified sea.

3. APPROXIMATE SOLUTIONS FOR THE SHALLOW STRATIFIED SEA

Let us introduce the following parameters: $T(z)$ is the transmittance of the water layer enclosed between the ocean surface and depth z; $R(z)$ is the diffuse reflection coefficient of the ocean layer enclosed between depth z and the bottom. Let us assume that the irradiance of a horizontal surface placed just below the sea surface is E_0. In this case we can express the downward and upward irradiances E_d and E_u through E_0, the transmittance $T(z)$ and the diffuse reflection coefficient $R(z)$:

$$E_d(z) = E_0 T(z), \quad E_u(z) = R(z) E_d(z) = E_0\, T(z)\, R(z). \tag{10}$$

After inserting Eqns. (10) to the system of Eqn. (3) and performing some algebra, we obtain the following equations for transmittance $T(z)$ and diffuse reflection coefficient $R(z)$:

$$\frac{dT(z)}{dz} + \left[2 - \bar{\mu}(z)\right]\left[a(z) + b_B(z)\right] T(z) = 0, \tag{11}$$

$$\frac{dR(z)}{dz} - 4\left[a(z) + b_B(z)\right] R(z) = -b_B(z)\left\{2 - \bar{\mu}(z) + R^2(z)\left[2 + \bar{\mu}(z)\right]\right\}. \tag{12}$$

The solution of Eqn. (11) with the boundary condition $T(0) = 1$ is obtained instantly:

$$T(z) = \exp\left\{-\int_0^z \left[2 - \bar{\mu}(z')\right]\left[a(z') + b_B(z')\right] dz'\right\}. \tag{13}$$

Eqn. (12) is of the Riccati type. For arbitrary functional dependencies $a(z)$ and $b_B(z)$ this type of equation cannot be solved in analytic form.

3.1. Homogeneous ocean.

For a homogeneous optically infinitely deep sea illuminated by diffuse light, $dR/dz = 0$ and a physical solution of Eqn. (12) will be $R = R_\infty$, where [4]

$$R_\infty = \left(\frac{1 - \bar{\mu}}{1 + \bar{\mu}}\right)^2, \quad \bar{\mu} = \frac{a}{\sqrt{a + 3b_B + \sqrt{b_B(4a + 9b_B)}}}. \tag{14}$$

Equation (14) is valid for any combination of inherent optical properties a and b_B and satisfies two limit relations: at $\bar{\mu} \to 0$ (pure scattering and non-absorbing medium) $R_\infty \to 1$; and at $\bar{\mu} \to 1$ (pure absorbing and nonscattering medium) $R_\infty \to 0$.

Eqn. (14) also can be represented in the form of Eqn. (1) with the coefficient k equal to

$$k = \frac{1 + 4\bar{\mu}^2 - \bar{\mu}^4}{(1+\bar{\mu})^4}, \quad 0.25 \le k \le 1. \tag{15}$$

It is interesting that for the typical value of mean cosine $\bar{\mu} \sim 0.67$ the coefficient $k \approx 0.33$. This value agrees with the value proposed by Prieur and Morel [12]. When $b_B / a << 1$ Eqn. (14) gives the value of R_∞ calculated in the single-scattering approximation, $R_\infty \cong b_B/(4a)$. The values of R_∞, computed using Eqn. (14) with the values of $\bar{\mu}$ typical to the World Ocean, are very close to the values of R_∞ obtained by Gordon and Brown [13].

Eqn. (12) also has another exact solution for a homogeneous shallow ocean with depth z_B and bottom albedo A_B:

$$R = R_\infty \frac{(R_0 - A_B) + (A_B - R_\infty)(R_0 R_\infty)^{-1} \exp(-4a z_B/\eta)}{(R_0 - A_B) + (A_B - R_\infty)\exp(-4a z_B/\eta)}, \tag{16}$$

where

$$\left.\begin{aligned} & R_0 = \frac{2+\bar{\mu}}{2-\bar{\mu}} R_\infty, \quad \eta = \frac{1-g}{(1-\bar{\mu}/2)/k - 1}, \quad g = \frac{(1-\bar{\mu}^2)^2}{1+\bar{\mu}^2(4-\bar{\mu}^2)}, \quad 0 \le \eta \le 1, \\ & \eta = \begin{cases} \frac{12}{7}\bar{\mu}\left(1 - \frac{13}{21}\bar{\mu}^2 + \frac{47}{147}\bar{\mu}^4 - \frac{185}{1029}\bar{\mu}^6 + o(\bar{\mu}^7)\right), & \bar{\mu} << 1, \\ 1-(1-\bar{\mu})^2 - \frac{3}{8}(1-\bar{\mu})^4 + \frac{1}{2}(1-\bar{\mu})^5 + o((1-\bar{\mu})^5), & (1-\bar{\mu}) << 1. \end{cases} \end{aligned}\right\} \tag{17}$$

3.2. Stratified ocean.

Let us obtain an approximate solution of Eqn. (12) for the inhomogeneous ocean. For open areas of the World Ocean and many littoral waters, $R^2(z) \le 5 \cdot 10^{-3}$, $\bar{\mu} << 1$ and the second term in the braces in the right-hand side of Eqn.(12) can be neglected. The result is a linear equation for R:

$$-\frac{dR(z)}{dz} + 4\left[a(z) + b_B(z)\right] R(z) = b_B(z)\left[2 - \bar{\mu}(z)\right]. \tag{18}$$

To solve this equation let us introduce an auxiliary variable,

$$x(z) = 4\int_0^z \left[a(z) + b_B(z)\right] dz. \tag{19}$$

By using transformation (19) in Eqn. (18) we obtain the following linear equation:

$$\frac{dR(x)}{dx} + R(x) = R_\infty(x), \tag{20}$$

where

$$R_\infty(x) = \left[\frac{1-\bar{\mu}(x)}{1+\bar{\mu}(x)}\right]^2 \approx \frac{2-\bar{\mu}(x)}{4} \frac{b_B(x)}{a(x)+b_B(x)}. \tag{21}$$

Here we have neglected terms of the orders of R_0^2 and R_∞^2. Let us seek a solution of Eqn.(20) in the form of partial plus general solutions: $R = R_p + R_g$, where $R_g = C_g \exp(-x)$. The partial solution is defined through the Green’s function $G(x)$ of Eqn. (20):

$$R_p(x) = \int G(x - x')R_\infty(x')\,dx'. \tag{22}$$

Here

$$G(x) = H(x)\exp(-x), \tag{23}$$

is the solution of the following equation:

$$\frac{dG(x)}{dx} + G(x) = \delta(x), \tag{24}$$

where $\delta(x)$ is the Dirac's delta function, and $H(x)$ is the Heaviside or step function [14]:

$$\delta(x) = \begin{cases} \infty, & x = 0, \\ 0, & x \neq 0, \end{cases} \quad \int_{-\infty}^{+\infty} \delta(x)\,dx = 1, \quad H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad \frac{d\,H(x)}{dx} = \delta(x). \tag{25}$$

Let us consider the sea with shallow bottom that reflects light according to the Lambert's law with the albedo A_B. In this case the boundary condition for $R(x)$ at the bottom depth $z = z_B$ is written as:

$$R(x[z = z_B]) = A_B. \tag{26}$$

Returning to the variable z and applying the boundary condition (26), we obtain the following solution to Eqn. (20):

$$\begin{aligned} R(z) = 4\int_z^{z_B} R_\infty(z')\exp\left\{-4\int_z^{z'}\left[a(z'') + b_B(z'')\right]dz''\right\}\left[a(z') + b_B(z')\right]\,dz' \\ + A_B\exp\left\{-4\int_z^{z_B}\left[a(z') + b_B(z')\right]dz'\right\}. \end{aligned} \tag{27}$$

Equation (27) defines the diffuse reflection coefficient of the vertically stratified shallow ocean measured at depth z . By setting $z = 0$. we obtain the following equation that defines the diffuse reflection coefficient of the inhomogeneous ocean of depth z_B:

$$R = 4\int_0^{z_B} R_\infty(z)\exp\left\{-4\int_0^{z}\left[a(z') + b_B(z')\right]dz'\right\}\left[a(z) + b_B(z)\right]\,dz + A_B\exp\left\{-4\int_0^{z_B}\left[a(z') + b_B(z')\right]dz'\right\}. \tag{28}$$

For the optically thick ocean $\left(\int_0^{z_B}\left[a(z) + b_B(z)\right]dz >> 1\right)$ Eqn. (28) can be simplified even further:

$$R_{(\infty)} = 4\int_0^{\infty} R_\infty(z)\exp\left\{-4\int_0^{z}\left[a(z') + b_B(z')\right]dz'\right\}\left[a(z) + b_B(z)\right]\,dz. \tag{29}$$

4. VALIDATION

Several assumptions are usually made in the development of any two-flux theory. In none of the known theories is it possible to write an analytically defined criterion for the validity of the relationships obtained. The accuracy of the results, therefore, used to be estimated by comparing them with numerical solutions of the modeling of the transfer phenomenon.

Now, let us compare our results with the results of Monte Carlo numerical computations of diffuse reflection coefficient $R_{(\infty)}$. Let us choose one of the least favorable cases of discontinuous stratification then compare the values of $R_{(\infty)}$ computed with Eqn. (29) with the exact numerical results obtained by Gordon and Brown [13]. In the paper [13] the Monte Carlo method is used to compute the diffuse reflection coefficient of a stratified ocean. The stratification, used by Gordon and Brown, [13] consists of two water layers. The thickness h and scattering coefficient b_1 of the upper layer are used as variable parameters. The lower semi-infinite layer has fixed inherent optical properties. The optical parameters taken for the lower layer are typical to the Sargasso Sea in the 530-nm wavelength band: $a = 0.062\ m^{-1}$ and $b/a = 0.66$. The same value of a and the same scattering phase function with $B = 0.0272$ is adopted for both layers.

Starting from Eqn. (29), we derive a relation for the diffuse reflection of the two-layer ocean. Let the upper layer of thickness h be described by the optical properties b_1, a_1, and B_1; and let the lower layer have the properties b_2, a_2, and B_2. By performing the integration in Eqn. (29), we have,

$$R_{(2)} = R_1 + (R_2 - R_1)\exp\left[-4h(a_1 + b_{B1})\right], \tag{30}$$

here

$$R_1 = \left[(1-\bar{\mu}_1)/(1+\bar{\mu}_1)\right]^2, \quad R_2 = \left[(1-\bar{\mu}_2)/(1+\bar{\mu}_2)\right]^2. \tag{31}$$

$$\bar{\mu}_1 \equiv \bar{\mu}(a_1, b_{B1}), \ \ \bar{\mu}_2 \equiv \bar{\mu}(a_2, b_{B2}). \tag{32}$$

In our case, $a_1 = a_2 = 0.062$ $b_{B1} = B_1 b_1 = 0.0272\ b_1$, $b_{B2} = B_2 b_2 = 0.0011 m^{-1}$ and $R_2 = 0.005$.

If we regard the values for $R_{(2)}$ taken from Ref. [13] as the exact ones, then the relative error of computations with Eqn. (30) does not exceed 15% for $\omega_0 \le 0.85$. In waters with single scattering albedo $\omega_0 \le 0.6$ the error of calculations made with the Eqns. (28)-(29) does not exceed 10%.

5. CONCLUSION

Equations (28) and (29) derived here can be used to compute the diffuse reflection coefficient of the arbitrary stratified ocean. The precision of computations based on these formulae is comparable to the precision of *in situ* measurements of the diffuse reflection coefficient (10-15%). These equations allow us to study the influence of the scattering layers in water on the spectral signatures of light ascending from the sea. They also can be used to calculate the radiance contrasts between differently stratified areas of the sea. Formulae (28–(29) also can serve as a basis for remote algorithms for restoration of the vertical profiles of the sea optical properties.

ACKNOWLEDGMENTS

The author thanks continuing support at the Naval Research Laboratory through the Littoral Optical Environment (LOE 6640-07) and Optical Oceanography (OO 73-5051-07) programs. This article represents NRL contribution NRL/PP/7331–97–0030.

REFERENCES

1. E. P. Zege, A. P. Ivanov, and I. L. Katsev, *Image Transfer through a Scattering Media*, Springer Verlag, Berlin, pp. 349, 1991.

2. N. G. Jerlov, *Marine Optics*, Elsevier Sci. Pub. Co., Amsterdam-Oxford-New York, 1976.

3. V. I. Haltrin, G. W. Kattawar, “Self-consistent solutions to the equation of transfer with elastic and inelastic scattering in oceanic optics: I. Model,” *Appl. Optics*, Vol. 32, No. 27, pp. 5356-5367, 1993.

4. V. I. Haltrin, “Theoretical and empirical phase functions for Monte Carlo calculations of light scattering in seawater,” in *Proceedings of the Fourth International Conference Remote Sensing for Marine and Coastal Environments*, Vol. I, Orlando, Florida, Publ. by Envir. Res. Inst. of Michigan, Ann Arbor, MI, pp. 509-518, 1997.

5. V. I. Haltrin, “Exact Solution of the Characteristic Equation for Transfer in the Anisotropically Scattering and Absorbing Medium,” *Appl. Optics*, Vol. 27, No., pp.599-602, 1988.

6. Timofeyeva, V. A., “Relation Between the Optical Coefficients in Turbid Media,” *Izvestiya USSR Acad. Sci., Atmos. Ocean Physics*, Vol. 8, pp. 654-656, 1972

7. I. S. Gradshteyn, and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, Inc., New York, 1160 pp., 1980.

8. V. I. Haltrin, “Algorithm for Computing Apparent Optical Properties of Shallow Waters under Arbitrary Surface Illumination,” in *Proceedings of the Third International Airborne Remote Sensing Conference and Exhibition* , Vol. I, Copenhagen, Denmark, Publ. by the Envir. Res. Institute of Michigan, Ann Arbor, MI, pp.463-470, 1997.

9. K. Schwarzschild, “Über das Gleichgewicht der Sonnenatmosphere,” *Gottingen Nachr.*, Vol. 41, pp. 1-24, 1906.

10. A. Schuster, “Radiation through a foggy atmosphere,” *Astrophys. J.*, Vol. 21, pp. 1-22, 1905.

11. P. Kubelka, and F. Munk. “Ein Beitrag zur Optik der Farbanstriche,” *Z. Techn. Phys*., Vol. 12, 593-607, 1931.

12. L. Prieur, and A. Morel. “Relations theoretiques entre le facteur de reflection diffuse de l’eau de mer.” *UGGI, XVI Ass. Gale*, Grenoble, pp. 278, 1975.

13. Gordon, H. R. and 0. B. Brown. “Diffuse reflection of the ocean: some effects of vertical structure,” *Appl. Optics*, Vol. 14, No. 12, pp. 2892-2895, 1975.

14. P. M. Morse, and Feshbach, H. *Methods of Theoretical Physics*, Parts 1 and 2, McGraw-Hill, New York, 1953.