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THEORETICAL AND EMPIRICAL PHASE FUNCTIONS FOR MONTE CARLO CALCULATIONS OF LIGHT SCATTERING IN SEAWATER*

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ABSTRACT

Simple empirical equations based on experimental measurements of phase functions which connect angular scattering coefficient and phase function with the inherent optical properties of seawater are proposed. The known theoretical solutions to the transfer equation in the asymptotic regime are organized for use as benchmark for Monte Carlo models. The equations and formulae presented here can be used for enhancing the precision and reducing the execution time of Monte Carlo calculations of light fields in seawater or any other type of scattering and absorbing media.

1. INTRODUCTION

Analysis of a number of publicly available and proprietary Monte Carlo codes which numerically simulate the light field in seawater and similar scattering and absorbing media shows that there is room for significant improvement of these codes. There is also a definite need in some objective criteria to test the correctness of these codes. One of the significant sources of error is the discretization of cumulative phase function as, for example, implemented in the code by Kirk (1981). This shortcoming can be eliminated or reduced if analytical or empirical equations are used to derive procedures for computing the scattering angle with any pre-specified precision.

Another reason for writing this paper is a need for some objective criterion of validation of the results of Monte-Carlo simulation. Theoretically it is possible to compute radiance distribution of the light in the depth of scattering media with an arbitrary scattering phase function. Since this computation is based on analytical relationships, it is very fast and gives much higher precision than usual Monte-Carlo technique. A few theoretical phase functions, which give the exact solution for radiance distribution in asymptotic regime, are also should be employed for such validation. All of them are listed with the relevant equations for parameters which can be incorporated in the Monte-Carlo algorithms. The theoretical phase functions include a Henyey-Greenstein, a delta-hyperbolic (Haltrin, 1988), and an elliptic one with the eccentricity ratio varying from one to infinity.

Most modelings of light propagation in the ocean use volume scattering functions of Petzold (1972). The robustness of these functions has never been adequately demonstrated. In addition to Petzold phase functions we also analyze the experimental phase functions of other investigators. These additional experimental phase functions include eight unpublished earlier Man'kovsky (1995) phase functions and a number of selected phase functions of other investigators quoted in Jerlov (1976). All analyzed phase functions are represented as a custom regressions which fits data with $r^2 \geq 0.99$. The eccentricity factors for experimental phase functions vary from less than 5 and up to 150.

The shapes of radiance distributions for each of the presented phase functions are either derived analytically or computed with a numerical procedure based on earlier work by Loskutov (1969). For some families of processed phase functions (particularly, Petzold's) the secondary regressions are determined. They connect the custom regression coefficients with the inherent optical properties of the water. All secondary regressions have $r^2 \geq 0.87$. The strong regressions give an empirical model of the phase functions with the coefficients dependent on the absorption and scattering coefficients, the single-scattering albedo varies over most of the range for natural water (0.09 to 0.96).

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2. THEORY

In the paper by Haltrin (1988) it was shown that the solution to the radiative transfer equation in the depth of scattering medium can be written as

$$L(z,\mu,\varphi) = L_0\,\Psi(\mu)\exp(-\gamma\tau)\,, \tag{1}$$

where $L(z,\mu,\varphi)$ is the radiance of light at depth z, L_0 is determined from the boundary conditions, $\psi(\mu)$ is a radiance distribution as a function of $\mu=\cos\theta$, θ and φ are zenith and azimuth angles, the optical depth is $\tau = cz$, where $c=a+b$ is the beam attenuation coefficient, a and b are the absorption and scattering coefficients, $\gamma = k/c$ is the eigenvalue of the asymptotic radiative transfer equation, k is the diffuse attenuation coefficient.

The radiance distribution $\Psi(\mu)$ in Eqn. (1) is the solution of the characteristic or asymptotic equation for radiative transfer

$$(1-\gamma\mu)\Psi(\mu) = \frac{\omega_0}{2}\int_{-1}^{1}\Psi(\mu')\,\bar{p}(\mu,\mu')\,d\mu'\,, \tag{2}$$

where $\omega_0 = b/c$ is the albedo for single scattering, $\mu' = \cos\theta'$, and

$$\bar{p}(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} p(\cos\chi)\,d\varphi,\quad \cos\chi = \mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\,\cos(\varphi-\varphi') \tag{3}$$

is the phase function averaged over azimuth, χ is the scattering angle, and the scattering phase function $p(\cos\chi)$ is normalized by the following condition

$$\frac{1}{2}\int_0^{\pi} p(\cos\chi)\sin\chi\,d\chi = 1\,. \tag{4}$$

If we represent the phase function in the form of the Legendre polynomial series

$$p(\cos\chi) = \sum_{n=0}^{\infty} s_n P_n(\cos\chi),\qquad s_0 = 1\,, \tag{5}$$

where $P_n(\cos\chi)$ is the Legendre polynomial of the *n*-th order, then we have

$$\bar{p}(\mu,\mu') = \sum_{n=0}^{\infty} s_n P_n(\mu)\,P_n(\mu')\,. \tag{6}$$

The Legendre polynomial series can also be used to represent the radiance distribution $\Psi(\mu)$:

$$\Psi(\mu) = \sum_{n=0}^{\infty}(2n+1)\,\psi_n P_n(\mu) \tag{7}$$

By substituting Eqns. (6) and (7) into Eqn. (2) and utilizing the recursion relation

$$(2n+1)\,\mu\,P_n(\mu) = n\,P_{n-1}(\mu) + (n+1)\,P_{n+1}(\mu) \tag{8}$$

we obtain relationships between the expansion coefficients ψ_n and s_n (see Chandrasekhar, 1960):

$$\left.\begin{array}{c} \psi_0 - \gamma\,\psi_1 = \omega_0\psi_0 s_0\,, \\ \gamma\,(n+1)\,\psi_{n+1} - (2n+1-\omega_0 s_n)\,\psi_n + \gamma\,n\,\psi_{n-1} = 0\,,\quad n = 1,\ldots,\infty\,. \end{array}\right\} \tag{9}$$

From the condition of solvability of the homogeneous system of equations (9), we can find the connection between γ, ω_0 and s_n. This condition, is simply that the determinant Δ of the system of equation (9):

$$\Delta = \begin{vmatrix} 1-\omega_0 & -\gamma & 0 & 0 & \ldots \\ -\gamma & 3-\omega_0 s_1 & -2\gamma & 0 & \ldots \\ 0 & -2\gamma & 5-\omega_0 s_2 & -3\gamma & \ldots \\ 0 & 0 & -3\gamma & 7-\omega_0 s_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}, \tag{10}$$

be equal to 0. As was shown in Loskutov (1969), an infinite chain fraction relates γ and ω_0:

$$1-\omega_0 = \cfrac{\gamma^2}{3-\omega_0 s_1 - \cfrac{4\gamma^2}{5-\omega_0 s_2 - \cfrac{9\gamma^2}{7-\omega_0 s_3 - \ldots}}}\,, \tag{11}$$

or

$$1-\omega_0 = \Delta_1 \tag{12}$$

where Δ_n can be obtained from

$$\Delta_n = \frac{(n\gamma)^2}{2n+1-\omega_0\, s_n - \Delta_{n+1}}, \quad n = 1, \ldots, \infty\,. \tag{13}$$

If we know the single scattering albedo ω_0 and the phase function coefficients s_n, the eigenvalue γ can be obtained numerically by solving Eqn. (12)-(13). In this case we compute all coefficients ψ_n using the following relationships:

$$\left.\begin{aligned} &\psi_0 = 1, \qquad \psi_1 = \frac{1-\omega_0}{\gamma}, \\ &\psi_n = \left(2 - \frac{1+\omega_0\, s_{n-1}}{n}\right)\frac{\psi_{n-1}}{\gamma} - \left(1-\frac{1}{n}\right)\psi_{n-2}\,,\ \ n = 2, \ldots, \infty, \end{aligned}\right\}, \tag{14}$$

and therefore restore the radiance distribution $\Psi(\mu)$ given by Eqn. (7).

This procedure can be used to retrieve the asymptotic irradiance distribution in a scattering and absorbing medium.

3. THEORETICAL PHASE-FUNCTIONS

For computational efficiency a theoretical or numerical representation of the phase function is most appropriate. These theoretical phase functions also can provide a means by which other techniques, such as Monte Carlo, may be evaluated.

3.1 The Henyey-Greenstein Phase Function

A very popular and frequently used phase function was proposed by Henyey and Greenstein (1941). It has the following functional form:

$$p_{HG}(\mu) = \frac{1-g^2}{\left(1+g^2-2g\mu\right)^{3/2}} \equiv \sum_{n=0}^{\infty}(2n+1)\,g^n P_n(\mu), \qquad 0 < g < 1\,. \tag{15}$$

The backscattering probability can be expressed as

$$B = \frac{1}{2}\int_{-1}^{0} p_{HG}(\mu)\,d\mu = \frac{1-g}{2g}\left(\frac{1+g}{\sqrt{1+g^2}} - 1\right) \tag{16}$$

and the cumulative phase function:

$$C_{HG}(\mu) = \frac{1}{2}\int_{\mu}^{1} p_{HG}(\mu)\,d\mu = \frac{1-g^2}{2g}\left(\frac{1}{1-g} - \frac{1}{\sqrt{1+g^2-2g\mu}}\right). \tag{17}$$

Reverse cumulative phase function, *i.e.* the solution of Eqn.(17) in respect to μ:

$$\mu = \frac{1}{2g}\left[1+g^2 - \left(\frac{1-g^2}{1+g-2g\,C_{HG}}\right)^2\right]. \tag{18}$$

Depth light distribution:

$$\psi_{HG}(\mu) = \sum_{n=0}^{\infty}(2n+1)\,\psi_n P_n(\mu) \tag{19}$$

The coefficients ψ_n in Eqn. (19) can be obtained from Eqn. (14) with $s_n = (2n+1)\,g^n$ and γ taken from Table 1.

Table 1. The eigenvalues γ of the asymptotic equation for transfer for the Henyey-Greenstein scattering function and different values of the single-scattering albedo ω_0 and parameter *g*.

$g \setminus \omega_0$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.85	0.90	0.95	0.99
0.50	0.9974	0.9776	0.9412	0.8911	0.8280	0.7511	0.6579	0.5418	0.4709	0.3856	0.2733	0.1224
0.55	0.9955	0.9707	0.9296	0.8754	0.8092	0.7301	0.6359	0.5207	0.4512	0.3683	0.2602	0.1162
0.60	0.9928	0.9626	0.9166	0.8585	0.7891	0.7079	0.6128	0.4986	0.4304	0.3501	0.2464	0.1097
0.65	0.9892	0.9532	0.9023	0.8401	0.7676	0.6843	0.5884	0.4751	0.4085	0.3308	0.2317	0.1027
0.70	0.9844	0.9424	0.8864	0.8201	0.7444	0.6591	0.5624	0.4502	0.3852	0.3102	0.2160	0.0952
0.75	0.9785	0.9299	0.8687	0.7982	0.7194	0.6319	0.5345	0.4234	0.3601	0.2881	0.1990	0.0871
0.80	0.9710	0.9155	0.8489	0.7740	0.6919	0.6023	0.5042	0.3944	0.3329	0.2639	0.1804	0.0781
0.85	0.9616	0.8987	0.8261	0.7467	0.6612	0.5695	0.4707	0.3624	0.3028	0.2371	0.1595	0.0680
0.90	0.9497	0.8783	0.7993	0.7149	0.6258	0.5318	0.4325	0.3258	0.2684	0.2064	0.1354	0.0560
0.95	0.9335	0.8518	0.7652	0.6750	0.5818	0.4855	0.3857	0.2812	0.2263	0.1686	0.1054	0.0407
0.97	0.9246	0.8379	0.7475	0.6546	0.5594	0.4620	0.3620	0.2587	0.2052	0.1496	0.0901	0.0326
0.99	0.9124	0.8190	0.7237	0.6272	0.5295	0.4308	0.3308	0.2291	0.1774	0.1246	0.0699	0.0214

3. 2 The Delta-Hyperbolic Phase-Function

Another theoretical phase function proposed by Haltrin (1988) is the delta-hyperbolic phase function. Its advantage is that it allows an exact analytic solution for the radiance distribution in the depth of scattering medium. This solution has the functional form of Henyey-Greenstein function. This phase function has the following form:

$$p_H(\mu) = 2g\,\delta(1-\mu) + \frac{1-g}{\sqrt{2(1-\mu)}} \equiv \sum_{n=0}^{\infty} (1+2ng)\,P_n(\mu), \quad 0 \le g \le 1, \tag{20}$$

where $\delta(1-\mu)$ is a Dirac delta function, and *g* is the shape parameter.

The backscattering probability is expressed through the shape parameter as

$$B = \frac{1}{2}\int_{-1}^{0} p_H(\mu)\,d\mu = \frac{1-g}{2+\sqrt{2}} \cong 0.2929\,(1-g)\,. \tag{21}$$

The cumulative phase function used in Monte Carlo calculations is:

$$C_H(\mu) = \frac{1}{2}\int_{\mu}^{1} p_H(\mu')\,d\mu' = \begin{cases} g + (1-g)\sqrt{\dfrac{1-\mu}{2}}, & \mu < 1, \\ 0, & \mu = 1. \end{cases} \tag{22}$$

Reverse cumulative phase function, *i.e.* the solution of Eqn.(22) in respect to μ::

$$\mu = \begin{cases} 1, & 0 \le C_H \le g, \\ 1 - 2\left(\dfrac{C_H - g}{1-g}\right)^2, & g < C_H \le 1, \end{cases} \tag{23}$$

and the eigenvalue of asymptotic equation for transfer (2) with the phase function (20) is:

$$\gamma = (1-\omega_0)/\eta\,, \tag{24}$$

The depth radiance distribution, *i.e.* the solution of Eqn. (2) with the phase function (20) is:

$$\psi(\mu) = \frac{1-\eta^2}{\left(1+\eta^2-2\eta\mu\right)^{3/2}}, \qquad \eta = \sqrt{\frac{1-\omega_0}{1+\omega_0-2g\,\omega_0}}\,. \tag{25}$$

3.3 Transport Phase Function

This theoretical phase function is widely used in neutron transport theory (Davison, 1957). It has the advantage of reducing the radiation transfer equation to the easily solvable case of isotropic scattering. This function has the following form:

$$p_T(\mu) = 2B + 2(1-2B)\,\delta(1-\mu) \tag{26}$$

where *B* is the backscattering probability.

The cumulative phase function is:

$$C_T(\mu) = \frac{1}{2}\int_{\mu}^{1} p_T(\mu')\,d\mu' = \begin{cases} 1 - B(1+\mu), & 0 \le \mu < 1 \\ 0, & \mu = 1 \end{cases} \tag{27}$$

The reverse cumulative phase function, *i.e.* the solution of Eqn.(27) in respect to μ::

$$\mu = \begin{cases} 1, & 0 \le C_T < 1 - 2B \\ \dfrac{1 - C_T}{B} - 1, & 1 - 2B < C_T \le 1 \end{cases} \tag{28}$$

The depth radiance distribution, *i.e.* the solution of Eqn. (2) with the phase function (26) is:

$$\psi(\mu) = \frac{2\,\omega_0\,B}{1 - \omega_0 + 2\,\omega_0\,B - \gamma\,\mu}, \tag{29}$$

where γ is the eigenvalue of asymptotic equation for transfer (2) with the phase function (26). It is easy to show that γ is a solution of the following equation:

$$\frac{\omega_0\,B}{\gamma}\ln\left(\frac{1 - \omega_0 + 2\,\omega_0\,B + \gamma}{1 - \omega_0 + 2\,\omega_0\,B - \gamma}\right) = 1\,. \tag{30}$$

Pre-calculated values of γ for different *B* and ω_0 are given in Table 2. The isotropic scattering case can be derived from the transport one with $B = 0.5$.

Table 2. The eigenvalues γ of the asymptotic equation for transfer for the transport scattering function (26) and different values of the single-scattering albedo ω_0 and the backscattering probability *B.*

B\ ω_0	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.97	0.99	0.999
0.10							.99997	.99995	.99993	.99992	.99991
0.15					.99985	.99952	.99878	.99819	.99790	.99758	.99743
0.20			.99910	.99952	.99840	.99605	.99194	.98909	.98778	.98637	.98570
0.25		.99991	.99933	.99741	.99316	.98562	.97398	.96639	.96299	.95939	.95770
0.30		.99952	.99741	.99195	.98150	.96472	.94040	.92504	.91825	.91107	.90771
0.35	.99985	.99840	.99316	.98150	.96122	.93041	.88708	.85995	.84795	.83526	.82931
0.40	.99952	.99605	.98562	.96472	.93041	.87973	.80866	.76355	.74335	.72177	.71157
0.45	.99878	.99195	.97398	.94040	.88708	.80866	.69566	.62031	.58519	.54643	.52758
0.49	.99775	.98709	.96122	.91471	.84170	.73274	.56631	.44226	.37730	.29542	.24877
0.50	.99741	..98562	.95750	.90733	.82864	.71041	.52543	.37949	.29638	.17251	.05475

4. EXPERIMENTAL PHASE FUNCTIONS

In deriving regressions for experimentally measured phase functions we used custom or manual approach, *i.e.* we tried to regress against a function of angle which gives the best possible fit.

4.1 Regressions for the Kopelevich phase functions

This phase function is based on *in situ* measurements and was proposed in a numerical form by Kopelevich (1983) as a part of his physical model of scattering in seawater. It expresses the total hydrosol scattering function as a linear combination of two phase functions, where p_s is a small particles scattering phase function which correspond to terrigenic suspensions, and p_l is a large particles scattering function which is associated with the biogenic fraction of marine hydrosol

$$p_K(\mu) = \frac{C_s\,p_s(\mu) + C_l\,p_l(\mu)}{C_s + C_l} \tag{31}$$

where C_s and C_l are the concentrations (*g/m*^{3}) of small and large particles respectively. The small- and large-component phase functions can be expressed by the following regressions:

$$p_S(\mu) = \exp\left(\sum_{n=0}^{5} s_n\,\theta^{3n/4}\right),\quad p_L(\mu) = \exp\left(\sum_{n=0}^{5} l_n\,\theta^{3n/4}\right),\quad \theta = \cos^{-1}(\mu) \tag{32}$$

where θ is the scattering angle in degrees and the coefficients s_n and l_n are given in Table 3.

The particulate scattering and backscattering coefficients associated with the phase functions (31) can be described as:

$$b^{(p)}(\lambda) \;=\; b_s^0(\lambda)\,C_s + b_l^0(\lambda)\,C_l\;, \tag{33}$$

$$b_B^{(p)}(\lambda) \;=\; B_s\,b_s^0(\lambda)\,C_s + B_l\,b_l^0(\lambda)\,C_l\;, \tag{34}$$

$$B_s = 0.5\int_{-1}^{0} p_s(\mu)\,d\mu = 0.039\,,\quad B_l = 0.5\int_{-1}^{0} p_l(\mu)\,d\mu = 6.4\cdot 10^{-4}\;, \tag{35}$$

here $b_s^0(\lambda)$ and $b_l^0(\lambda)$ are specific scattering coefficients for the small and large particulate matter respectively, B_s is the backscattering probability for the small particles, and B_l is the backscattering probability for the large particles. The specific scattering coefficients for small and large particulate are:

$$b_s^0(\lambda) \;=\; \left(1.1513\,m^2/g\right)\left(\lambda_0/\lambda\right)^{1.7},\quad b_l^0(\lambda) \;=\; \left(0.3411\,m^2/g\right)\left(\lambda_0/\lambda\right)^{0.3},\quad \lambda_0 = 400\,nm\,. \tag{36}$$

Table 3. The coefficients of regression for two basic Kopelevich phase functions p_s and p_l.

n	0	1	2	3	4	5	r^2
s_n	1.725880	-2.957089E-2	-2.782943E-2	1.255406E-3	-2.155880E-5	1.356632E-7	0.999
l_n	5.238466	-1.604327	8.157686E-2	-2.150389E-3	2.419323E-5	-6.578550E-8	0.999

4.2. Regressions for the Petzold phase functions

The Petzold scattering phase functions (1972) are used very extensively in Monte Carlo simulations. We processed them in order to obtain regressions suitable for fast numerical computations.

4.2.1. Regressions for individual phase functions:

The angular scattering coefficient can be expressed as a following regression:

$$\beta(\theta) = \exp\left[\sum_{n=0}^{5} (-1)^n s_n\,\theta^{\frac{n}{2}}\right], \tag{37}$$

with the scattering coefficient derived by formula:

Table 4. Regression coefficients for fifteen Petzold phase functions and accompanying inherent optical properties.

#	a	b	ω_0	B	s_0	s_1	s_2	s_3	s_4	s_5	r^2
01	0.082	0.117	0.588	0.025	7.1701	6.2714	1.6186	0.232770	1.5870E-2	3.9957E-4	0.999
02	0.114	0.037	0.247	0.044	5.2747	5.4289	1.3814	0.204610	1.4460E-2	3.7545E-4	0.999
03	0.122	0.043	0.258	0.038	5.7010	5.7448	1.4830	0.219030	1.5367E-2	3.9623E-4	0.999
04	0.195	0.275	0.585	0.014	8.1107	6.0057	1.4594	0.204070	1.3531E-2	3.2752E-4	0.999
05	0.179	0.219	0.551	0.013	7.7330	5.8914	1.4518	0.207600	1.4057E-2	3.4736E-4	0.999
06	0.337	1.583	0.824	0.019	10.309	6.0403	1.4271	0.191980	1.2366E-2	2.9508E-4	0.999
07	0.366	1.824	0.833	0.020	10.642	6.2615	1.5264	0.211720	1.4115E-2	3.5115E-4	0.999
08	0.125	1.205	0.906	0.018	9.7922	6.0099	1.4784	0.209020	1.4147E-2	3.5554E-4	0.999
09	0.093	0.009	0.093	0.119	1.7357	3.2121	0.6276	0.091713	7.0079E-3	1.9620E-4	0.999
10	0.138	0.547	0.798	0.018	6.3606	3.2975	0.6113	0.087507	6.2367E-3	1.5826E-4	0.999
11	0.764	0.576	0.430	0.017	6.4660	3.3303	0.6247	0.091382	6.6720E-3	1.7430E-4	0.999
12	0.196	1.284	0.867	0.015	9.3499	5.1115	1.0851	0.140410	8.8762E-3	2.0729E-4	0.999
13	0.188	0.407	0.685	0.017	8.3200	5.7313	1.3783	0.192000	1.2780E-2	3.1384E-4	0.999
14	0.093	0.081	0.463	0.025	6.6031	5.8843	1.4927	0.219070	1.5361E-2	3.9730E-4	0.999
15	0.085	0.008	0.091	0.146	0.5526	2.8685	0.6899	0.117600	9.5531E-3	2.7590E-4	0.996

$$b = 2\pi \int_0^{\pi} \beta(\theta) \sin\theta \, d\theta \, . \tag{38}$$

Then the scattering phase function is represented by the regression:

$$p(\theta) = \frac{4\pi}{b} \exp\left[\sum_{n=0}^{5} (-1)^n s_n \, \theta^{\frac{n}{2}} \right]. \tag{39}$$

The coefficients of regression s_n in Eqns. (37) and (39) are given in Table 4.

4.2.2. Regressions for all fifteen Petzold phase functions.

An empirical representation of the Petzold's experimental angular scattering coefficient $\beta(\theta)$ and of the scattering phase function $p(\theta)$, where θ is the scattering angle in degrees (r^2 is the regression coefficient) are represented by the following equations:

$$\beta(\theta) = \exp\left[q \left(1 + \sum_{n=1}^{5} (-1)^n k_n \, \theta^{\frac{n}{2}} \right) \right] \tag{40}$$

$$q = 2.598 + 17.748\sqrt{b} - 16.722\, b + 5.932\, b\sqrt{b}, \qquad r^2 = 0.996 \tag{41}$$

$$\left. \begin{aligned} k_1 &= 1.188 - 0.688\,\omega_0, & r^2 &= 0.925, \\ k_2 &= 0.1(3.07 - 1.90\,\omega_0), & r^2 &= 0.897, \\ k_3 &= 0.01(4.58 - 3.02\,\omega_0), & r^2 &= 0.893, \\ k_4 &= 0.001(3.24 - 2.25\,\omega_0), & r^2 &= 0.887, \\ k_5 &= 0.0001(0.84 - 0.61\,\omega_0), & r^2 &= 0.870, \end{aligned} \right\} \tag{42}$$

$$p(\theta) = \frac{4\pi}{b} \exp\left[q \left(1 + \sum_{n=1}^{5} (-1)^n k_n \, \theta^{\frac{n}{2}} \right) \right] \tag{43}$$

$$\frac{1}{2} \int_0^{\pi} p(\theta) \, \sin\theta \, d\theta = 1 \tag{44}$$

The strong regressions given by Eqns. (40)-(44) can be used as a basis for the empirical model of the phase functions with the coefficients dependent on the absorption and scattering coefficients. The single-scattering albedo used here varies from 0.09 to 0.96.

4.3 Regressions for the Man'kovsky phase functions

We also used a set of eight phase functions measured by V. I. Man'kovsky (1995) in different locations of World Ocean. They can be represented empirically as:

$$p_{Ma}^{(i)}(\mu) = 10^{\sum_{n=0}^{4} m_n \theta^{n/2}}, \quad \theta = \cos^{-1}(\mu) \tag{45}$$

The coefficients m_n are given in Table 5.

Table 5. Coefficients m_n for the Man'kovsky phase functions (45) b and B.

#	c	b	B	m_0	m_1	m_2	m_3	m_4	r^2
1	0.557	0.410	0.00781	2.866737	-1.649911	0.2008073	-1.654026E-2	6.278219E-4	0.992
2	0.403	0.299	0.01429	2.260724	-1.316699	0.1533971	-1.325010E-2	5.234336E-4	0.997
3	0.276	0.157	0.01786	2.089051	-1.361259	0.1400787	-9.357579E-3	3.149432E-4	0.995
4	0.161	0.106	0.02128	2.943783	-2.436226	0.4008064	-3.366019E-2	1.108607E-3	0.986
5	0.143	0.062	0.03704	2.301892	-2.056033	0.3246083	2.729402E-2	9.187083E-4	0.989
6	0.626	0.518	0.01370	1.933749	-0.882802	0.0625397	-5.860951E-3	3.135987E-4	0.997
7	0.435	0.290	0.02041	1.572035	-0.788899	0.0345644	-2.310001E-3	1.716351E-4	0.998
8	0.585	0.467	0.01010	2.949902	-1.771696	0.2619379	-2.292908E-2	8.054979E-4	0.997

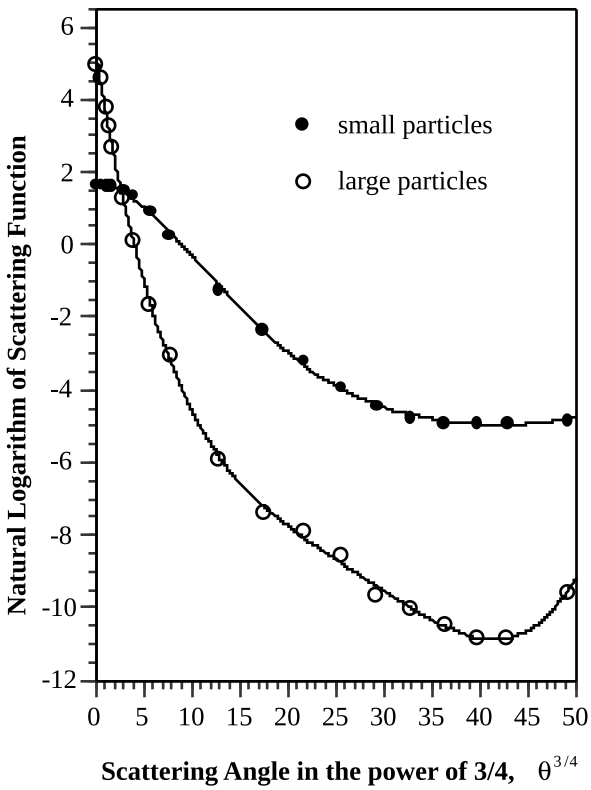

Fig. 1. The components of the Kopelevich phase function.

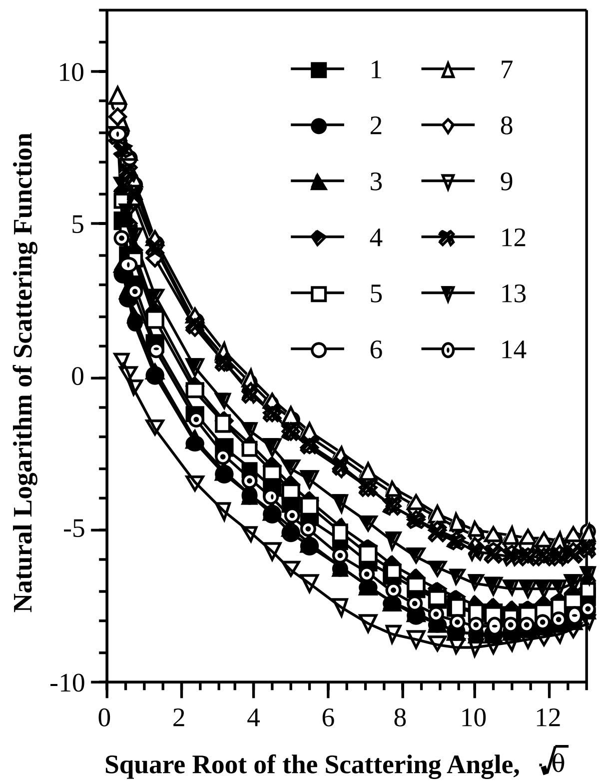

Fig. 2. The Petzold phase functions.

4.4. Regressions for other experimental phase functions

There is a number of phase functions cited in the book by Jerlov (1960). They are measured by Hulburt, Atkins, Kozlyaninov, Spielhaus, Ochakovsky, Sasaki, Jerlov, Tyler, Duntley and Kullenberg (see references in Jerlov). All of them can be represented as the following regressions:

$$p_{Je}(\mu) = \exp\left(\sum_{n=0}^{5} e_n \, \theta^{n/2}\right), \quad \theta = \cos^{-1}(\mu)\,. \qquad (46)$$

with the coefficients given in Table 6.

Table 6. Regression coefficients for phase functions given in Jerlov (1960).

#	Author \ n ->	0	1	2	3	4	5	r^2
1	Hulburt	20.06327	-10.31779	2.854634	-4.132459E-1	2.826287E-2	-7.178163E-4	0.999
2	Atkins	8.61641	-0.826512	-0.075622	7.030181E-3	-	-	0.999
3	Kozlyaninov	10.90142	-2.192057	0.229901	-2.062461E-2	8.271728E-4	-	0.999
4	Spielhaus	11.85671	-1.987467	0.059177	2.070856E-3	-	-	0.998
5	Ochakovsky	12.02006	-3.210226	0.460431	-3.991154E-2	1.397485E-3		0.996
6	Sasaki, 1	4.51332	0.708010	-0.260644	1.424003E-2	-	-	0.994
7	Sasaki, 2	8.75572	-1.233455	-0.090305	4.517135E-3	-	-	0.987
8	Jerlov	13.76925	-5.039381	1.271560	-1.948416E-1	1.419847E-2	-3.768597E-4	0.999
9	Tyler	11.16772	-1.801671	0.044614	2.361708E-3	-	-	0.998
10	Duntley	11.34007	-1.991247	0.083241	-1.709973E-4	-	-	0.999
11	Kullenberg, 1	16.88578	-7.772059	1.920117	-2.643937E-1	1.790696E-2	-4.586527E-4	0.996
12	Kullenberg, 2	9.37208	-0.568721	-0.250431	3.645848E-2	-1.988283E-3	4.776097E-5	0.999

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Publication by Environmental Research Institute of Michigan (ERIM), Ann Arbor, Michigan 48113-4001, USA, 1997.

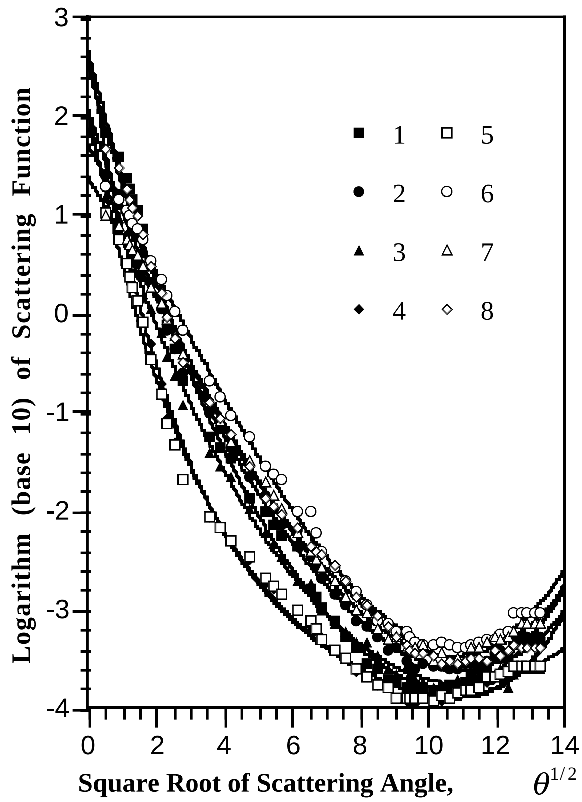

Fig.3 The Man'kovsky phase functions.

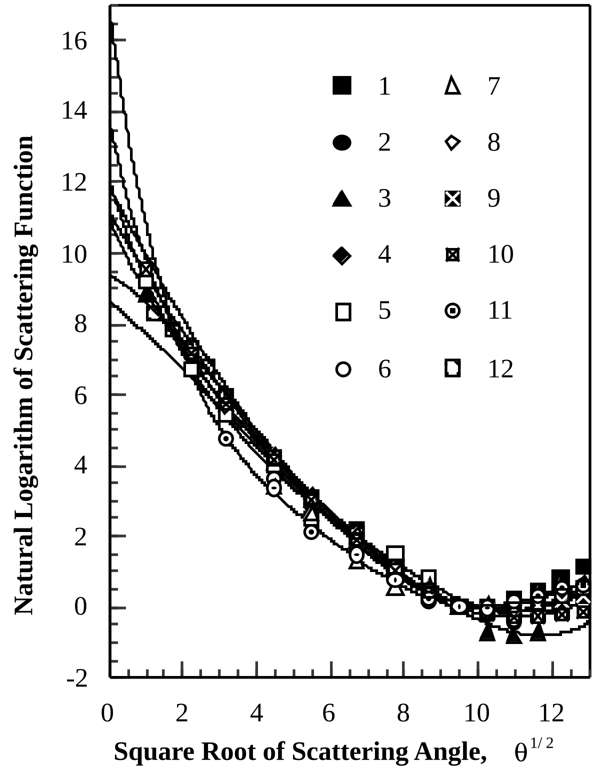

Fig.4 The phase functions cited in Jerlov.

5. CONCLUSIONS

Several theoretical and more than thirty experimentally measured phase functions were analyzed and processed here in order to make them suitable for use in enhancing and benchmark testing of Monte Carlo calculations of light fields in seawater. The theoretical phase functions included a Henyey-Greenstein, a delta-hyperbolic (Haltrin), and a transport one with the eccentricity ratio varying from one to infinity. Experimental phase functions measured by Petzold, Man'kovsky and a number of other investigators quoted in Jerlov are represented as simple customly derived regressions which fits data with $r^2 \geq 0.99$. The eccentricity factors for the experimental phase functions presented here (*i.e.* the ratio of forward to backward scattering probabilities) vary from less then 5 and up to 150.

The simple empirical equations which connect the volume scattering coefficient and the phase function with inherent optical properties of seawater are derived. The known theoretical solutions of the transfer equation in the asymptotic regimein a form useful for benchmark testing of Monte Carlo calculations are presented . The equations and formulae presented here can be used for enhancing precision and reducing the execution time of Monte Carlo computations of light fields in marine water or any other type of scattering media.

For some families of processed phase functions the secondary regressions are determined. They connect the custom regression coefficients with the inherent optical properties of the water. All secondary regressions have regression coefficients $r^2 \geq 0.87$. The strong regressions give an empirical model of the phase functions with the coefficients dependent on the absorption and scattering coefficients. The single-scattering albedo used in these regressions varies over most of the range for natural water (0.09 to 0.96).

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