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Radiance Distribution of Sunlight Reflected from a Shadowed Sea*

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ABSTRACT

Results for the actual radiance distribution of light are obtained numerically using the *in situ* measured optical properties of seawater. The scattering phase function is estimated through the regression equations that are based on all fifteen Petzold scattering phase functions. The derived scattering phase function depends the scattering coefficient, the single-scattering albedo, and the scattering angle. The final result is an expression for the upwelling sea angular radiance distribution as a function of: the sun zenith angle, azimuthal and zenith angles of viewing, the width of the opaque body, absorption and attenuation coefficients, and the phase function of scattering. The resulting radiance distributions are discussed and illustrated.

1.0 INTRODUCTION

Detailed quantitative interpretations of images sometimes require data about the angular distribution of light reflected from a shadowed sea. We derive the solution of the two-dimensional radiative transfer problem for the sea shadowed by an opaque body such as a pier. The solution of the problem requires two different theoretical approaches: one for the depth dependence and one for the horizontal field. The solution for the depth dependence is obtained using Green's function theory (Vladimirov, 1971). The solution in the horizontal direction is obtained with the one-dimensional Fourier transform method.

The algorithm is based on the analytical solution of the spatially two-dimensional (Cartesian coordinates x and z) scalar radiative transfer equation in seawater. The $0x$ axis starts on the sea surface in the center of the pier shadow and increases towards the sun in the direction orthogonal to the axis of the pier. The $0z$ axis increases down in the direction of the sea bottom.

For the estimation of the scattering phase function we use the regression equations that are based on the processing of all fifteen Petzold phase functions (Haltrin, 1997). These relationships connect the scattering phase functions with the scattering coefficient and the single-scattering albedo.

The final equation expresses upwelling sea radiance distribution as a function of the sun zenith angle, azimuthal and zenith angles of viewing, the width of the pier and the inherent optical properties of the water (absorption and attenuation coefficients and the phase function of scattering). Results for the actual radiance distribution of light are obtained numerically using the *in situ* measured optical properties of seawater. The scattering phase function is estimated through the regression equations that are based on all fifteen Petzold scattering phase functions. The derived scattering phase function depends the scattering coefficient, the single-scattering albedo, and the scattering angle.

The main final result is an expression for the upwelling sea angular radiance distribution as a function of: 1) the sun zenith angle, 2) azimuthal and zenith angles of viewing, 3) the width of the opaque body, 3) absorption and attenuation coefficients, and 4) the phase function of scattering. The resulting radiance distributions are illustrated.

The future development of this problem may consist of the following enhancements to the theory: consideration of the reflection by the sea bottom; consideration of the atmospheric scattering effects and the elevation of the pier; consideration of the sea surface roughness; generalization to the non-infinite pier (a 3D-problem in space); consideration of the vertical optical inhomogeneity of seawater; and enhancement of the part of the theory related to the higher-order scattering effects.

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2.0 APPROACH

We formulate our goal as follows: Given a downwelling radiance distribution from the sun consistent with a long rectangular shadow, model the water-leading radiance as a function of azimuthal and zenith angles of viewing and distance in a direction normal to the long dimension of the rectangle. We do the above for given solar elevation.

Several serious attempts have been made to solve two- and three-dimensional radiative transfer problems (Vladimirov, 1961; Williams, 1967; Kochetkov, 1968; Kaper, 1969; Case and Hazeltine, 1970, Flateau and Stephens, 1988). Unfortunately none of them are simple and convenient enough to produce practically usable equations applicable to solve our problem.

The origin of Cartesian coordinates is placed on the sea surface just below the center of the pier. For simplicity we place the pier along 0y axis perpendicular to the sun plane. The 0x axis is directed to the sun, and 0*z* axis is directed to the bottom.

We start from the scalar radiative transfer theory (Chandrasekhar, 1960). We have a two dimensional in space radiative transfer problem. The origin of Cartesian coordinates is placed on the sea surface just below the center of an infinite along 0*y* axis pier. The 0z coordinate is directed to the bottom of the sea. The 0*x* coordinate is orthogonal to the pier.

We use Green's function method (Vladimirov, 1971) to solve the problem on z-direction, and Fourier transform method to solve x-dependent part (Morse and Feshbach, 1953).

The radiance angular distribution is split into three components: Unscattered, single-scattered and multiple scattered components. Exact solutions for the first two components is found. An approximate solution to the third, multiple scattered component, is found.

3.0 HOMOGENEOUS SHALLOW SEA ILLUMINATED BY SUNLIGHT

In this paper we follow standard hydrooptics notation (Jerlov, 1986; Mobley, 1994).

Let us introduce the following notations: E_S is irradiance by the sun on the sea surface, $E_S^0 = E_S / \mu_S^0$ is irradiance by the sun on the surface normal to the sun rays, $\mu_S^0 = \cos Z_S \equiv \sin h_S$ is the cosine of zenith angle Z_S, $h_S = 90° - Z_S$ is a sun elevation angle.

The sun radiance above the sea surface will be

$$L_S^0 = E_S^0\, \delta(\mu - \mu_S^0)\, \delta(\varphi), \tag{1}$$

here $\delta(x)$ is a Dirac's delta-function (Morse and Feshbach, 1953), and

$$\mu_S = \sqrt{1 - \sin^2 Z_S / n_w^2} \tag{1a}$$

is a cosine of penetration angle with $n_w \cong 4/3$ as the seawater refraction coefficient.

If $T_S^{\downarrow} = 1 - R_F^{\downarrow}(Z_S)$ is a transmission of sea surface, where R_F is a Fresnel reflection coefficient from above, than the radiance below the sea surface is:

$$L_w^0(\mu,\varphi) = E_w^0\, \delta(\mu - \mu_S)\, \delta(\varphi), \quad E_w^0 = E_S^0\, T_S^{\downarrow}, \tag{2}$$

here E_w^0 is the irradiance by the sun on the surface normal to the rays just below the sea surface.

The one-dimensional scalar radiative transfer equation (Chandrasekhar, 1960) for the total radiance is:

$$\left(\mu \frac{d}{dz} + c\right) L(z,\mu,\varphi) = \frac{b}{4\pi} \int_0^{2\pi} d\varphi' \int_{-1}^{1} p(\cos\gamma)\, L(z,\mu',\varphi')\, d\mu', \tag{3}$$

here $c = a + b$ is the attenuation coefficient, a is the absorption coefficient and b is the scattering coefficient of seawater, $\mu = \cos\theta$, the direction of light propagation is determined by the zenith and azimuth angles θ and φ, $p(\cos\gamma)$ is the scattering phase function, γ is the scattering angle determined from the following formula:

$$\cos\gamma = \mu\mu' + \sqrt{1-\mu^2}\, \sqrt{1-\mu'^2}\, \cos(\varphi - \varphi'). \tag{4}$$

Let us represent total radiance of light inside the sea L as the direct light radiance L_0, and the sum of the radiances L_n that represent n^{th} order of scattering:

$$L = \sum_{n=0}^{\infty} b^n L_n \, . \qquad (5)$$

Substitution of Eq. (5) into Eq. (3) gives us the following equations for radiance components:

$$\left(\mu \frac{d}{dz} + c \right) L_0(z, \mu, \varphi) = 0 \, , \qquad (6)$$

$$\hat{D} L_n(z,\mu,\varphi) = b \hat{S} L_{n-1}(z,\mu,\varphi), \text{ or } L_n(z,\mu,\varphi) = b \hat{D}^{-1} \hat{S} L_{n-1}(z,\mu,\varphi) = b \hat{T} L_{n-1}(z,\mu,\varphi). \qquad (7)$$

Operators $\hat{D}$, $\hat{S}$ and $\hat{T}$ are defined as follows:

$$\left. \begin{aligned} \hat{D} F(z,\mu,\varphi) &\equiv \left(\mu \frac{d}{dz} + c \right) F(z,\mu,\varphi), \\ \hat{S} F(z,\mu,\varphi) &\equiv \frac{1}{4\pi} \int_0^{2\pi} d\varphi' \int_{-1}^{1} p(\cos\gamma) F(z,\mu',\varphi') \, d\mu', \end{aligned} \right\} \qquad (8)$$

$$\hat{D}^{-1} \equiv \hat{G}, \quad \hat{T} = \hat{D}^{-1} \hat{S} \equiv \hat{G} \hat{S} .$$

Operator $\hat{G}$ is a Green's function of Eq. (3). It is defined according to the following formula (Vladimirov, 1971):

$$\hat{G} F(z) \equiv \int_{-\infty}^{+\infty} G(z - z') F(z') \, dz', \quad G(z) = \frac{H(z/\mu)}{|\mu|} e^{-cz/\mu}, \qquad (9)$$

Here $H(z)$ is the Heavyside or step function $(H(z) = 1, \ z \geq 0; \ H(z) = 0, \ z < 0)$.(Morse and Feshbach, 1953). The total scattering operator is:

$$\hat{T} F(z,\mu,\varphi) \equiv \frac{1}{4\pi} \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\mu' \, p(\cos\gamma) \int_{-\infty}^{+\infty} dz' G(z - z') F(z',\mu',\varphi') \; . \qquad (10)$$

Now, the total radiance is expressed as a following sum:

$$L = \sum_{n=0}^{\infty} b^n L_n \, , \quad L_n = b^n \hat{T}^n L_0, \ n \geq 1, \ L(z,\mu,\varphi) = \sum_{n=0}^{\infty} b^n \hat{T}^n L_0(z,\mu,\varphi). \qquad (11)$$

Let us start solving these equations.

4.0 SOLUTIONS FOR THE RADIANCE

The solution for direct (unscattered) radiance L_0 is straightforwardly obtained from the Eq. (6) with the following boundary condition on the sea surface, $L_0(0,\mu,\varphi) = L_w^0(\mu,\varphi)$:

$$L_0(z,\mu,\varphi) = E_w^0 \, e^{-cz/\mu_S} \delta(\mu - \mu_S) \delta(\varphi). \qquad (12)$$

Equation (7) for the single-scattered radiance can be written as:

$$L_1(z,\mu,\varphi) \equiv \frac{1}{4\pi} \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\mu' \, p(\cos\gamma) \int_{-\infty}^{+\infty} dz' G(z - z') L_0(z',\mu',\varphi') \; . \qquad (13)$$

By substituting Eq. (12) into Eq. (13), we have the following solution for single-scattered radiance distribution:

$$L_1(z,\mu,\varphi) = \frac{E_w^0 \, p(\cos\gamma_S)}{4\pi \, c |\mu|} \left[\psi_1(z,\mu) H(\mu) + \psi_2(z,\mu) H(-\mu) \right], \qquad (14)$$

here

$$\psi_1(z,\mu) = \frac{e^{-cz/\mu_S} - e^{-cz/\mu}}{1/\mu - 1/\mu_S}, \quad \psi_1(z,\mu_S) = c \, z \, e^{-cz/\mu_S}, \quad \psi_1(z,\mu)\big|_{z\to 0} = c \, z \, , \qquad (15)$$

$$\psi_2(z,\mu) = \frac{e^{-cz/\mu_S}}{1/\mu_S + 1/|\mu|} \quad . \qquad (16)$$

For a precision about 20% it is sufficient (see Shuleykin) to include only the next term in expansion given by Eq. (11). The explicit solution for the next term can be obtained from the following equation:

$$L_2(z,\mu,\varphi) \equiv \frac{1}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} d\mu'\, p(\cos\gamma) \int_{-\infty}^{+\infty} dz' G(z-z')\, L_1(z',\mu',\varphi') \ . \tag{17}$$

with L_1 given by Eqns. (14)-(16) and $G(z)$ by Eq. (9). The right boundary conditions has no influence on upwelling radiance distribution, so we do not discuss them here.

3.1 SOLUTIONS FOR THE UPWELLING RADIANCE NEAR THE SEA SURFACE

Taking integrals in (17) and summing all required terms at $z = 0$ and $\mu < 0$, we have the following solution for the case of homogeneous illumination:

$$L(0,\mu,\varphi) = \frac{E_w^0\,\mu_S\, b}{4\pi\, c\left(\mu_S + |\mu|\right)}\left[p(\cos\gamma_S) + \frac{b}{2\,c}\psi_p(\mu,\varphi)\right], \quad \mu < 0\,, \tag{18}$$

here

$$\cos\gamma_S = \mu\mu_S + \sqrt{1-\mu^2}\,\sqrt{1-\mu_S^2}\,\cos\varphi\,, \tag{19}$$

$$\psi_p(\mu,\varphi) = |\mu|\int_0^1 \frac{x_p(\mu,\mu',\varphi)}{\mu' + |\mu|}\,d\mu' + \frac{\mu_S}{\mu_S + |\mu|}\int_{-1}^{0} x_p(\mu,\mu',\varphi)\,d\mu'\,, \tag{20}$$

$$x_p(\mu,\mu',\varphi) = \frac{1}{2\pi}\int_0^{2\pi} p(\cos\gamma)\, p(\cos\gamma_S')\, d\varphi'\,, \tag{21}$$

$$\cos\gamma_S' = \mu'\mu_S + \sqrt{1-\mu'^2}\,\sqrt{1-\mu_S^2}\,\cos\varphi'\,. \tag{22}$$

4.0 HOMOGENEOUS SEA ILLUMINATED BY SUNLIGHT WITH SHADOW

Next we formulate a two-dimensional problem that takes into account unhomogeneous over horizontal axis $0x$ illumination. The two-dimensional radiative transfer equation for the total angular radiance distribution is:

$$\left(\mu\frac{\partial}{\partial z} + \sqrt{1-\mu^2}\,\cos\varphi\frac{\partial}{\partial x} + c\right)L(x,z,\mu,\varphi) = \frac{b}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} p(\cos\gamma)\, L(x,z,\mu',\varphi')\, d\mu'\,, \tag{23}$$

By defining a one-dimensional Fourier transform by the following formulae:

$$f_k = \int_{-\infty}^{+\infty} f(x)\, e^{-ikx} dx\ , \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_k\, e^{ikx} dk\ . \tag{24}$$

The Fourier transform of Eq. (23) reduces it to a one-dimensional equation for the Fourier amplitude $L_k(z,\mu,\varphi)$:

$$\left(\mu\frac{d}{dz} + \dot{c}\right)L_k(z,\mu,\varphi) = \frac{b}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} p(\cos\gamma)\, L_k(z,\mu',\varphi')\, d\mu'\,, \tag{25}$$

here

$$\dot{c} = c\,(1 + ik\tau), \quad \tau = \sqrt{1-\mu^2}\,\cos\varphi \big/ c\,. \tag{26}$$

Similarly, Eq. (25) may be obtained from the Eq.(3) by replacing the real extinction coefficient c by the complex value $\dot{c}$ given by Eq. (26). It means that the solution of Eq. (24) can be obtained from the solution (18) by replacing all x-dependent values by their Fourier transforms and all instances of the extinction coefficient c by the $\dot{c}$. Taking this into account, we have the following solution to Eq. (25):

$$L_k(0,\mu,\varphi) = \frac{E_{wk}\,\mu_S\, b}{4\pi\,\dot{c}\left(\mu_S + |\mu|\right)}\left[p(\cos\gamma_S) + \frac{b}{2\,\dot{c}}\psi_p(\mu,\varphi)\right]. \tag{27}$$

Now we only need to calculate a Fourier transform E_{wk} of the x-dependent radiance distribution

$$E_{wk} = \int_{-\infty}^{+\infty} E_w(x)\, e^{-ikx} dx \,. \tag{28}$$

In our case the sun illumination incorporates a shadowing at $-w \le x < w$, where w is a half width of a shadow. The angular-space distribution of undersurface illumination is:

$$L_w^0(\mu,\varphi) = E_w(x)\,\delta(\mu - \mu_S)\,\delta(\varphi)\,, \tag{29}$$

$$E_w(x) = E_w^0\left[1 - H(w - |x|)\right] \equiv E_w^0\left\{1 - 0.5\left[\mathrm{sign}(w - x) + \mathrm{sign}(w + x)\right]\right\}, \tag{30}$$

here $\mathrm{sign}(x) = |x|/x$, $i.e.$, $\mathrm{sign}(x) = 1,\ \ x > 0$, $\mathrm{sign}(x) = -1,\ \ x < 0$.

The Fourier transform of the surface illumination is:

$$E_{wk} = E_w^0\left[2\pi\,\delta(k) - (2/k)\sin(wk)\right]. \tag{31}$$

The x-dependent radiance distribution just below the sea surface is:

$$L(x,0,\mu,\varphi) = \frac{\mu_S\,\omega_0\,E_w^0}{4\pi\left(\mu_S + |\mu|\right)} \int_{-\infty}^{+\infty} \frac{\delta(k) - \sin(wk)/(k\pi)}{(1 + ik\tau)} \left[p(\cos\gamma_S) + \frac{\omega_0}{2\,(1 + ik\tau)}\psi_p(\mu,\varphi)\right] e^{ikx} dk \,. \tag{32}$$

By taking appropriate integrals we have the following solution for the undersurface upwelling radiance distribution:

$$L(x,0,\mu,\varphi) = \frac{\omega_0\,\mu_S\,E_w^0}{4\pi\left(\mu_S + |\mu|\right)} \left\{\left[1 - F_1(w,x,\tau)\right]\, p(\cos\gamma_S) + \frac{\omega_0}{2}\left[1 - F_2(w,x,\tau)\right]\psi_p(\mu,\varphi)\right\}, \tag{33}$$

$$\begin{aligned} F_1(w,x,\tau) &= \frac{1}{2}\left[\mathrm{sign}(w + x)\left(1 - \exp - \left|\frac{w + x}{\tau}\right|\right) + \mathrm{sign}(w - x)\left(1 - \exp - \left|\frac{w - x}{\tau}\right|\right)\right] \\ &+ \frac{1}{2}\,\mathrm{sign}\,\tau\left(\exp - \left|\frac{w - x}{\tau}\right| - \exp - \left|\frac{w + x}{\tau}\right|\right), \qquad \tau = \frac{\sqrt{1 - \mu^2}\,\cos\varphi}{c}\ , \end{aligned} \tag{34}$$

$$\begin{aligned} F_2(w,x,\tau) &= F_1(w,x,\tau) + \frac{1}{2\tau}\left[|w - x| \exp - \left|\frac{w - x}{\tau}\right| - |w + x| \exp - \left|\frac{w + x}{\tau}\right|\right] \\ &- \frac{1}{2|\tau|}\left[(w - x)\exp - \left|\frac{w - x}{\tau}\right| + (w + x)\exp - \left|\frac{w + x}{\tau}\right|\right], \end{aligned} \tag{35}$$

Far from the shadow in each direction (at $|w - x|,\ |w + x| >> |\tau|$) $F_1 \to 0$ and $F_2 \to 0$ and Eq. (33) coincides with the equation for upwelling radiance (18) for homogeneous illumination.
Equations (33)-(40) include functions ψ_p and x_p, given by Eqns. (20)-(21), that involve integrations over angular variables μ and φ. In order to simplify these expressions let us substitute phase functions inside integrals in Eqns (20)-(21) by their transport equivalents:

$$p(\cos\gamma) \to 2B + 2\,(1 - 2B)\,\delta(\mu - \mu')\,\delta(\varphi - \varphi')\,, \tag{36}$$

$$p(\cos\gamma_S') \to 2B + 2\,(1 - 2B)\,\delta(\mu' - \mu_S)\,\delta(\varphi' - \varphi_S)\,, \quad \varphi_S = 0\,, \quad B = 0.5\int_{-1}^{0} p(\mu)\,d\mu\,. \tag{37}$$

In this case after appropriate integrations we have:

$$x_p(\mu,\mu',\varphi) = 2B\,\overline{p}(\mu,\mu') + 2\,(1 - 2B)\,p(\cos\gamma_S)\,, \tag{39}$$

$$\overline{p}(\mu,\mu') = (1/2\pi)\int_0^{2\pi} p(\cos\gamma)\,d\varphi'\,, \tag{40}$$

$$\psi_p(\mu,\varphi) = 4B(1 - 2B)|\mu|\log\left(1 + 1/|\mu|\right) + 4\left[B(1 - B)\,\mu_S + (1 - 2B)^2|\mu|\right]\Big/\left(\mu_S + |\mu|\right). \tag{41}$$

5.0 RADIANCE ABOVE THE SEA SURFACE

In order to calculate radiance distribution above the sea surface as a function of viewing angles, we have to do the following: a) take into account the transmission by sea-air surface from below by multiplying result by the transmission coefficient:

$$T_S^{\uparrow}(\mu) = 1 - R_F^{\uparrow}(\mu); \tag{42}$$

b) express cosine $|\mu|$ through the zenith viewing angle θ:

$$|\mu| = \sqrt{1 - \sin^2\theta / n_w^2}\,; \tag{43}$$

This means that we should make the following two substitutions in Eq. (33):

$$E_w^0 \to E_S^0\, T_S^{\downarrow}(Z_S)\, T_S^{\uparrow}(\mu) \equiv E_S \left[1 - R_F^{\downarrow}(Z_S)\right]\left[1 - R_F^{\uparrow}(\theta)\right] \Big/ \cos Z_S\,, \quad \mu \to \sqrt{1 - \sin^2\theta / n_w^2}\,, \tag{44}$$

here $R_F^{\downarrow}$ and $R_F^{\uparrow}$ are Fresnel reflection coefficients of sea surface from above and below:

$$R_F^{\downarrow}(\mu) = \frac{1}{2}\left[\left(\frac{\mu - n_w\mu_w}{\mu + n_w\mu_w}\right)^2 + \left(\frac{n_w\mu - \mu_w}{n_w\mu + \mu_w}\right)^2\right],\ \ \mu_w = \frac{\sqrt{n_w^2 - 1 + \mu^2}}{n_w}, \tag{46}$$

$$R_F^{\uparrow} = \frac{1}{2}\left[\left(\frac{n_w\cos\theta - \mu_a}{n_w\cos\theta + \mu_a}\right)^2 + \left(\frac{\cos\theta - n_w\mu_a}{\cos\theta + n_w\mu_a}\right)^2\right],\ \ \mu_a = \sqrt{n_w^2 \sin^2\theta - 1}\,. \tag{47}$$

6.0 APPROXIMATION FOR THE PHASE FUNCTION OF SCATTERING

The backscattering coefficient is connected with the average cosine by the following equations (Haltrin, 1985, Haltrin and Kattawar, 1993):

$$b_B = \frac{a\,g}{1-g}, \quad g = \frac{(1-\bar{\mu}^2)^2}{1 + 4\bar{\mu}^2 - \bar{\mu}^4}, \tag{48}$$

This regression is derived by the author from the experimental results by Timofeyeva (1971)

$$\begin{aligned}\bar{\mu} &= y(2.6178398 + y(-4.6024180 + y(9.0040600 + y(-14.59994 + \\ &+ y(14.83909 + y(-8.117954 + 1.8593222\,y))))),\ \ y = \sqrt{1-\omega_0} \equiv \sqrt{b\ c}\,,\end{aligned} \tag{49}$$

An empirical representation of the Petzold's experimental angular scattering coefficient $\beta(\theta)$ and of the scattering phase function $p(\theta)$, where θ is the scattering angle in degrees, are represented by the following equations (Haltrin, 1997):

$$\beta(\theta) = \exp\left[q\left(1 + \sum_{n=1}^{5}(-1)^n k_n \theta^{\frac{n}{2}}\right)\right], \quad q = 2.598 + 17.748\sqrt{b} - 16.722\,b + 5.932\,b\sqrt{b} \tag{50}$$

$$\left.\begin{aligned} &k_1 = 1.188 - 0.688\,\omega_0\,, \quad k_2 = 0.1\left(3.07 - 1.90\,\omega_0\right), \quad k_3 = 0.01\left(4.58 - 3.02\,\omega_0\right), \\ &k_4 = 0.001\left(3.24 - 2.25\,\omega_0\right), \quad k_5 = 0.0001\left(0.84 - 0.61\,\omega_0\right), \end{aligned}\right\} \tag{52}$$

$$p(\theta) = \frac{4\pi}{b}\exp\left[q\left(1 + \sum_{n=1}^{5}(-1)^n k_n \theta^{\frac{n}{2}}\right)\right] \tag{53}$$

The strong regressions given by Eqns. (50)-(53) can be used as a basis for the empirical model of the phase functions with the coefficients dependent on the absorption and scattering coefficients. The single-scattering albedo used here varies from 0.09 to 0.96.

7.0 EXAMPLES OF COMPUTATIONS

To illustrate presented theory and algorithm, two examples of the sea shadowed by a long pier are presented here. The values of inherent optical properties are taken as follows; $c = 0.5\,m^{-1}$, $c = 0.5\,m^{-1}$. Figure 2 shows two-dimensional density plots of normalized sea radiance as a function of coordinate *x* and azimuth angle of viewing. The zenith angle of viewing is 30°, the solar zenith angle is 45°.

Fig. 3 shows the same reflected from the sea radiance as previous figures only for different angles: the zenith angle of viewing is 75°, and the solar zenith angle equal to 45°.

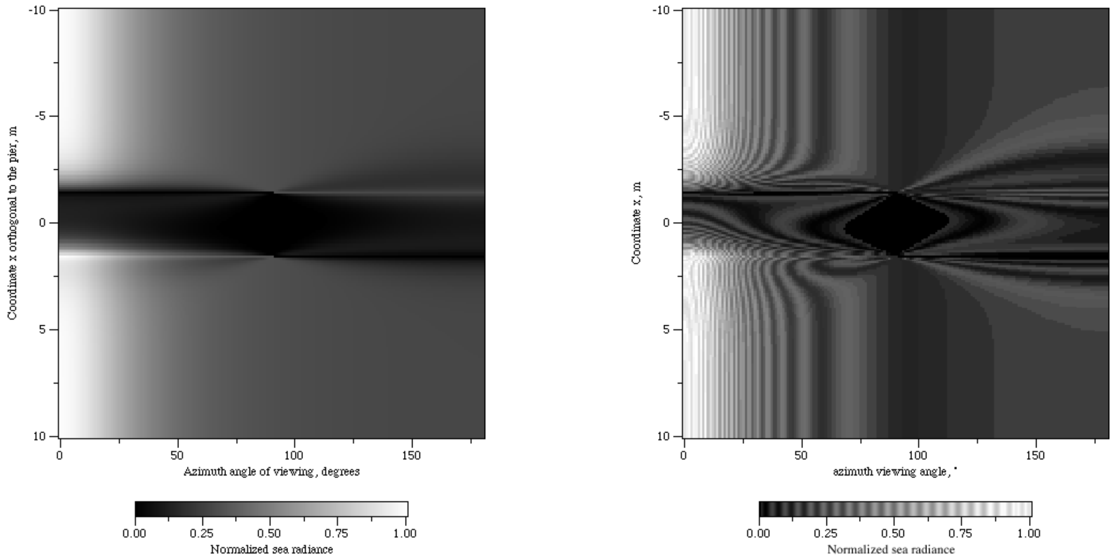

Fig. 2. Left: Two-dimensional density plot of Normalized sea radiance as a function of coordinate *x* and azimuth angle of viewing. Zenith angle of viewing is 30°, Sun zenith angle is 45°. Right: The same as left only plotted with the banded gray-scale palette to show the fine structure of the light field.

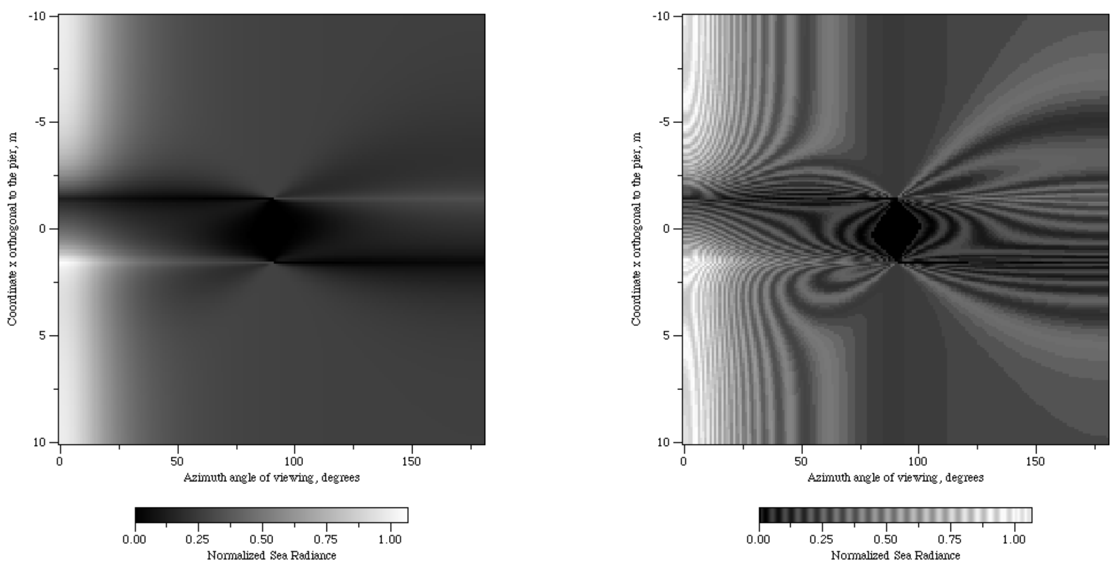

Fig. 3. Left: Two-dimensional density plot of Normalized sea radiance as a function of coordinate *x* and azimuth angle of viewing. Zenith angle of viewing is 75°, Sun zenith angle is 45°. Right: The same as left only plotted with the banded gray-scale palette to show the fine structure of the light field.

8.0 CONCLUSIONS

The final equations with E_w^0 given by Eq. (44) expresses upwelling sea radiance distribution as a function of the sun zenith angle, azimuthal and zenith angles of viewing, the width of the pier and the inherent optical properties of the water (absorption and attenuation coefficients and the phase function of scattering). Results for the actual radiance distribution of light are obtained numerically using the *in situ* measured optical properties of seawater and shown in Figs. 2–3. The scattering phase function used in these calculation is estimated through the regression equations (50)–(53) that are based on all fifteen Petzold scattering phase functions. The derived scattering phase function depends the scattering coefficient, the single-scattering albedo, and the scattering angle..

The main final result consists of the expressions (33)-(35) for the upwelling sea angular radiance distribution as a function of: the sun zenith angle, azimuthal and zenith angles of viewing, the width of the opaque body, absorption and attenuation coefficients, and the phase function of scattering.

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10.0 REFERENCES

S. Chandrasekhar, *Radiative Tranfer*, Dover Publications, New York, pp. 393, 1960.

V. V. Vladimirov, “Mathematical problems of the one-velocity theory of particle transport,” in: *Proc. V. A. Steklov Inst. of Math.*, Vol. 61, p. 3-158, 1961 (in Russian).

H. G. Kaper, “Elementary solutions of the reduced three-dimentional transport equation,” *J. Math. Phys.*, Vol. 10, No. 2, pp. 286-297, 1969.

K. M. Case and R. D. Hazeltine, “Three-Dimentional Linear transport theory,” *J. Math. Phys.*, Vol. 11, No. 4, pp. 1126-1135, 1970.

V. M. Kochetkov, “Fundamental solution of an unsteady transport equation in solid geometry,” *Appl. Math. Mech. (PMM)*, Vo. 32, No. 3, pp. 393-400, 1968.

P. J. Flateau and G. L. Stephens, “On the Fundamental solution of the radiative transfer equation,” *J. Geophys. Res.*, Vol. 93, No. D9, pp. 11037-11050, 1988.

M. M. Williams, “Approximate solutions of the neutron transport equation in two and three dimensional systems,” *Nucleonic*, Vol. 9, No. 7, pp. 305-312, 1967.

N. G. Jerlov, *Marine Optics*, Elsevier Press, Amsterdam, 231 pp, 1986.

C. D. Mobley, *Light and Water*, Academic Press, San Diego - Toronto, 592 p.,1994.

V. I. Haltrin and G. W. Kattawar “Self-consistent solutions to the equation of transfer with elastic and inelastic scattering in oceanic optics: I. Model,” *Appl. Optics*, Vol. 32, No. 27, pp. 5356-5367, 1993.

V. I. Haltrin (*a.k.a.* В. И. Халтурин), “Propagation of Light in a Sea Depth,” Chapter 2 in: *Remote Sensing of the Sea and the Influence of the Atmosphere* (in Russian), Moscow-Berlin-Sevastopol, Publ. by the GDR Academy of Sciences Institute for Space Research, pp. 20-62, 1985.

V. I. Haltrin, “Theoretical and empirical phase functions for Monte Carlo calculations of light scattering in seawater,” in *Proceedings of the Fourth International Conference Remote Sensing for Marine and Coastal Environments: Technology and Applications*, Vol. I, Publ. by Envir. Res. Inst. of Michigan, Ann Arbor, MI, pp. 509-518, 1997.

P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part 1 and 2, McGrawHill Book Co., New York-Toronto-London, 1978 pp.,1953.

V. V. Vladimirov, *Equations of Mathematical Physics*, Dekker, New York, p. 418, 1971.

V. V. Shuleykin, *Physics of the Sea* (in Russian), Nauka, Moscow, 1083 pp., 1968.

V. A. Timofeyeva, “Optical characteristics of turbid media of the seawater type,” *Izv., Atm. Ocean. Physics*, Vol.7, No.12, pp.1326-1329 (English translation: pp.863-865), 1971.