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An analytic Fournier-Forand scattering phase function as an alternative to the Henyey-Greenstein phase function in hydrologic optics

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Abstract – This work presents results of numerical analysis of the Fournier-Forand scattering phase function as an alternative to the Henyey-Greenstein phase function in hydrological optics. A number of equations are derived that connect different integral parameters of the Fournier-Forand phase function, including the normalization coefficient, with the parameters of this function. The Mathematica and FORTRAN programs, that computes the Fournier-Forand scattering phase function and some of its integral characteristics, are released for public use.

INTRODUCTION

The Henyey-Greenstein (HG) phase function, that was originally proposed for use in astrophysics [1], has become very popular in hydrologic optics, including some practically important underwater target detection algorithms. The HG phase function is appealing because it is very simple and convenient for mathematical analysis.

Other analytical phase functions that have been proposed for radiative transfer calculations are: 1) A three-parameter analytic phase function, a combination of forward and backward elongated HG phase functions proposed by G. W. Kattawar [2]. 2) A Haltrin phase function, a combination of hyperbolic and delta- functions [3], that gives an analytic asymptotic solution to the radiative transfer equation in the form of Henyey-Greenstein function with the elongation parameter depending on the inherent optical properties of the medium. 3) A Cornette-Shanks phase function [4] that converges to the Rayleigh phase function when the average cosine $\langle\mu\rangle << 1$ and approaches the HG phase function when $1-\langle\mu\rangle << 1$. 4) A Reynolds-McCormick two parameter phase function that generalizes the Henyey-Greenstein phase function [5] to a hyperbolic one.

When used in ocean optics all listed phase functions with all their advantages have one major shortcoming: their shapes don't resemble at all the shapes of realistic marine phase functions [6].

The two-parametric analytic Fournier-Forand (FF) scattering phase function was proposed in 1994 to the ocean optics community [7]. It was almost unnoticed by the optical oceanographic community possibly because the FF phase function has a more complex analytic form and the original paper did not include analysis of its properties. The major advantages of the Fournier-Forand phase function are: 1) it depends only on two parameters; and 2) it approximates almost all realistic marine phase functions with a very high degree of precision.

ORIGINAL FORMULA

The two-parametric analytic Fournier-Forand scattering phase function has the following form [7, 8]:

$$p(\mu) = \frac{A(1+\mu^2)}{(1-\delta)^2\delta^w}\left\{\left[w(1-\delta)-1+\delta^w\right]+ \frac{2}{(1-\mu)}\left[1-\delta^{w+1}-(1+w)(1-\delta)\right]\right\}, \tag{1}$$

where A is a normalization factor, and

$$w = \frac{3-\nu}{2}, \quad \delta = \frac{2(1-\mu)}{3(n-1)^2}, \; 3.5 \le \nu \le 5, \tag{2}$$

here $\mu = \cos\theta$, θ is a scattering angle. The two parameters of this phase function are: 1) n, relative to water refraction index of scattering particles, and 2) ν, a Junge parameter in the size distribution of particles used for derivation of Eq.(1). The phase function (1) is normalized according to the following rule:

$$0.5\int_{-1}^{1} p(\mu)\, d\mu = 1. \tag{3}$$

Expressions for several parameters of the FF phase function are given in the next section. They include: the average cosine,

$$\langle\mu\rangle = 0.5\int_{-1}^{1} p(\mu)\,\mu\, d\mu, \tag{4}$$

the backscattering probability,

$$B = 0.5\int_{-1}^{0} p(\mu)\, d\mu, \tag{5}$$

the average scattering angle,

$$<\theta> = 0.5\int_{-1}^{1} p(\mu)\cos^{-1}(\mu)\, d\mu, \tag{6}$$

and the average square of the scattering angle,

$$<\theta^2> = 0.5\int_{-1}^{1} p(\mu)\left[\cos^{-1}(\mu)\right]^2 d\mu. \tag{7}$$

INTEGRAL CHARACTERISTICS

The normalization factor A in Eq. (1), and parameters B, $\langle\mu\rangle$, $\langle\theta\rangle$ and $\langle\theta^2\rangle$ were calculated using the Mathematica code given in the APPENDIX 1. The normalization constant A is represented in the form of the following regression:

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$$A = 0.75 - 10^3 \sum_{i=0}^{5} \sum_{j=0}^{4} A_{ij}\, n^i (5-\nu)^{j+1},\ \ 1.05 \le n \le 1.35, \quad (8)$$

The calculated regression coefficients A_{ij} in Eq. (8) are given in Table 1.

Table 1. Coefficients A_{ij}

i\j	0	1	2	3	4
0	1.293678	-4.258598	5.5057687	-3.236333	0.7166757
1	-5.160532	17.06135	-22.11056	13.01506	-2.884615
2	8.251644	-27.36716	35.534350	-20.94020	4.644448
3	-6.605571	21.96185	-28.55899	16.84607	-3.738655
4	2.645947	-8.814722	11.47718	-6.775466	1.504461
5	-.4241213	1.415284	-1.844753	1.089783	-.2420895

The average cosine is represented as the following regression:

$$\langle \mu \rangle = 10^3 \sum_{i=0}^{5} \sum_{j=0}^{4} \mu_{ij}\, n^i (5-\nu)^{j+1},\ \ 1.05 \le n \le 1.35, \quad (9)$$

and the coefficients μ_{ij} are given in the following table:

Table 2. Coefficients μ_{ij}

i\j	0	1	2	3	4
0	4.542778	-16.149 42	21.97835	-13.33975	3.012829
1	-18.05841	64.43356	-87.87126	53.40034	-12.07039
2	28.79086	-102.9830	140.6605	-85.56511	19.35336
3	-22.98634	82.37450	-112.6488	68.57979	-15.51987
4	9.184525	-32.96302	45.12221	-27.48815	6.223457
5	-1.468693	5.277602	-7.230356	4.407131	-.9981772

The backscattering probability is given by the following formula:

$$B = 0.5 - 10^3 \sum_{i=0}^{5} \sum_{j=0}^{5} B_{ij}\, n^i (5-\nu)^{j+1},\ \ 1.05 \le n \le 1.35, \quad (10)$$

with the coefficients B_{ij} listed in the following table:

Table 3. Coefficients B_{ij}

i\j	0	1	2	3	4	5
0	2.586503	-11.93376	22.65763	-21.74635	10.40416	-1.968405
1	-10.29278	47.68255	-90.75762	87.24571	-41.78420	7.910808
2	16.42547	-76.30225	145.5101	-140.0586	67.13501	-12.71781
3	-13.12475	61.09552	-116.6888	112.4357	-53.93323	10.22203
4	5.247974	-24.46960	46.79444	-45.12891	21.66076	-4.107163
5	-.839769	3.920766	-7.505852	7.244199	-3.478885	.6598903

The average scattering angle is given by the expression:

$$\langle \theta \rangle = \frac{\pi}{2} - 10^3 \sum_{i=0}^{5} \sum_{j=0}^{5} \theta_{ij}\, n^i (5-\nu)^{j+1},\ \ 1.05 \le n \le 1.35, \quad (11)$$

with the coefficients θ_{ij} in the following table:

Table 4. Coefficients θ_{ij}

i\j	0	1	2	3	4	5
0	8.353436	-33.33324	59.32662	-55.27668	26.04694	-4.884773
1	-33.31039	133.4762	-238.1457	222.2226	-104.8146	19.66926
2	53.23549	-213.9708	382.5100	-357.3793	168.7010	-31.67553
3	-42.58630	171.5842	-307.2300	287.3467	-135.7372	25.49844
4	17.04415	-68.81035	123.3753	-115.4945	54.59060	-10.25924
5	-2.729399	11.03787	-19.81365	18.56251	-8.778586	1.650377

The average square of the scattering angle is represented as:

$$\langle \theta^2 \rangle = \frac{9\pi^2 - 34}{18} - 10^3 \sum_{i=0}^{5} \sum_{j=0}^{5} \vartheta_{ij}\, n^i (5-\nu)^{j+1}, 1.05 \le n \le 1.35, (12)$$

with the coefficients ϑ_{ij} given in Table 5.

Table 5. Coefficients ϑ_{ij}

i\j	0	1	2	3	4	5
0	16.08031	-71.17296	132.3328	-125.7178	59.83410	-11.28704
1	-64.02228	284.5727	-530.4455	504.7193	-240.4577	45.39026
2	102.2007	-455.6267	850.9605	-810.7228	386.5685	-73.01290
3	-81.68180	364.9879	-682.7602	651.1666	-310.7106	58.71421
4	32.66634	-146.2389	273.9218	-261.4816	124.8450	-23.60175
5	-5.227705	23.43964	-43.95440	41.99072	-20.05925	3.793589

ILLUSTRATIONS

The angular distributions of the Fournier-Forand scattering function are calculated using the FORTRAN code given in the APPENDIX 2. Several examples of calculated FF phase functions are shown in Figure 1. All calculated examples strikingly resemble experimental phase functions measured by different authors and presented in Ref. [6].

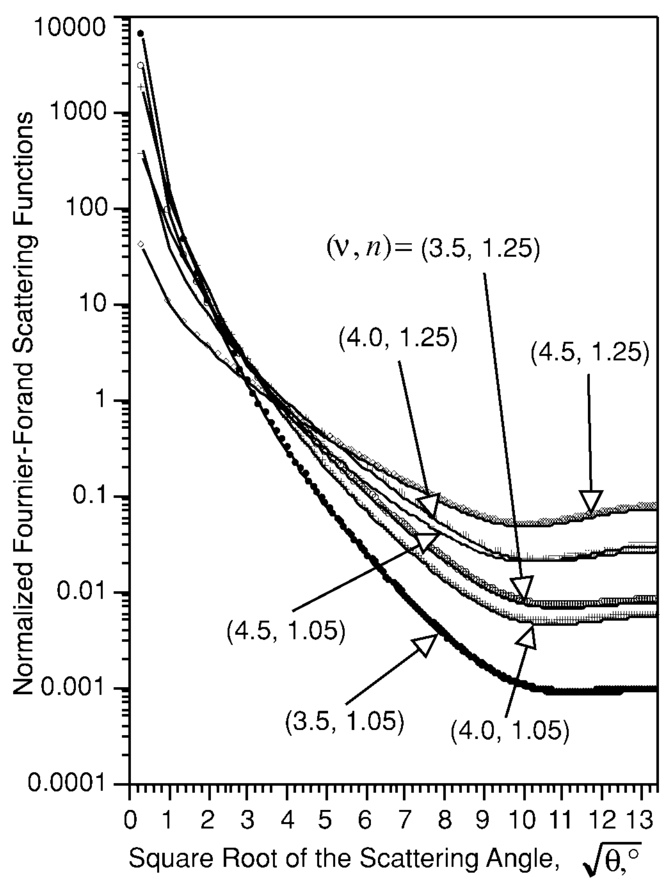

Figure 1. Examples of normalized Fournier-Forand phase functions for a set of parameters (ν, n).

CONCLUSION

Our model calculations confirm conclusion of Ref. [7] that in approximating experimental oceanic phase functions the Fournier-Forand phase function is preferable to the phase functions listed in the INTRODUCTION. Consequently it is more suitable for using in radiative transfer models in seawater.

ACKNOWLEDGMENT

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APPENDIX 1: MATHEMATICA CODE TO CALCULATE INTEGRAL PARAMETERS OF THE FOURNIER-FORAND PHASE FUNCTION

```
n=1.25;
nu =4.0;

al=1/(3*(n-1)^2);
u[mu_]:=Sqrt[2*(1-mu)];
fu[mu_]:=4/(u[mu])^2;
d[mu_]:=al*u[mu]*u[mu];
ed[mu_]:=1-d[mu];
dv[mu_]:=(d[mu])^v;
w[mu_]:=v*ed[mu]-1+dv[mu]+fu[mu]*
                    (1-d[mu]*dv[mu]-(1+v)*ed[mu]);
p0[mu_]:=0.5*(1+mu*mu)*w[mu]/(ed[mu]*ed[mu]*dv[mu]);
A0=N[1/NIntegrate[p0[mu],{mu,-1,1}],10];
p[mu_]:=0.5*A0*(1+mu mu*w[mu]/
            (ed[mu]*ed[mu]*dv[mu]);

B=N[NIntegrate[p[mu],{mu,-1,0}],10];
muAv=N[NIntegrate[mu*p[mu],{mu,-1,1}],10];
thAv=N[NIntegrate[ArcCos[mu]*p[mu],{mu,-1,1}],10];
thSqAv=N[NIntegrate[ArcCos[mu]*ArcCos[mu]*p[mu],
           {mu,-1,1}],10];

Print["  A0  = ",A0]
Print["  B   = ",B]
Print[" <µ> = ", muAv]
Print[" <ø>  = ", thAv]
Print["<ø^2> = ", thSqAv]
```

APPENDIX 2: A FORTRAN CODE TO CALCULATE THE FOURNIER-FORAND PHASE FUNCTION OF SCATTERING

```
! ***********************************************
      real  function ffphf(nu,n,thet,A0)
! ***********************************************
!   Computes a normalized Fournier-Forand phase
!   function of scattering. This code is based on
!   the Eq.(1) of this  paper, i. e. on the cor-
!   rected [9] version of the original equation
!   by G.R.Fournier and J.L.Forand [7]. A0 should
!   be precomputed with the code given above or
!   with the Eq.(8) of this paper (less precise).
! ===============================================
      implicit  none
      real      nu,n,thet,A0, v,ev,dn,d2
      real      mu,u2,u,delta,ed,dv,s1,s2,px

      v = 0.5*(3.-nu)
      ev = 1.+v
      dn = n-1.
      d2 = dn*dn
      mu = COSD(thet)
      u2 = 2*(1.-mu)
      u = SQRT(u2)
      delta = u2/(3.*d2)
      ed = 1.-delta
      dv = delta**v
      s1 = v*ed-1.+dv
      s2 = 1.-delta*dv-ev*ed
      s2 = 4*s2/u2
      px = s1+s2
      s2 = 8*ed*ed*dv
      ffphf = A0*px*(1.+mu*mu)/s2

      return
      end
! ***********************************************
```

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[8] An original formula in Refs. [7] contains misprint in the denominator: $(1-\delta^2)$ should be replaced by $(1-\delta)^2$.

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